

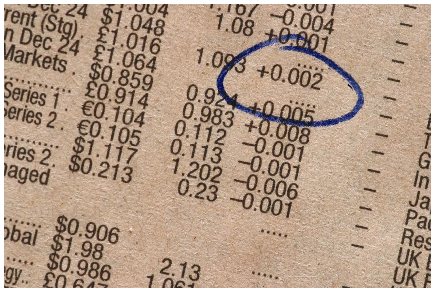

Teaching and Learning

Guide 8:

Partial Differentiation

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Section 1: Introduction to the guide

In economics, we are often concerned with analysing situations where a dependent variable is driven by two or more independent variables. This guide is designed to help lecturers set out the basic mathematical concepts and techniques needed for their students to effectively learn how to do economic analysis in such situations. The basic concepts covered are: partial derivatives, unconstrained optimisation, Lagrange multipliers and constrained optimisation. One challenge for economics lecturers is to ensure that all students develop a good understanding and also to ensure that those who are mathematically gifted do get a richer and deeper insight.

A large part of this guide then is designed to offer practical strategies to help students achieve at economics by improving their grasp of the fundamental methods of constrained optimisation – a technique which is at the heart of understanding individual choice of economic agents in a multitude of different settings.

This guide focuses on the ‘mathematical fundamentals’ which students need to undertake constrained optimisation. These mathematical fundamentals translate into several skills or competencies which will mean students will have:

- A good working knowledge of the relevant core mathematical concepts and tools;
- An ability to apply these concepts and tools to economic issues and problems;
- The capacity to think independently and critically about these concepts and know which concepts and methods to apply; and
- An ability to evaluate the usefulness of the mathematical concepts and appreciate that mathematical tools cannot always yield perfect or complete solutions to contemporary economic problems.
- An ability to understand that economic intuition is often helpful in short circuiting the full mathematical solution to a problem.

The guide could be used to inform a scheme of work or to devise strategies to deliver mathematical concepts. Some topics could be put easily together – such as bi-variate functions, partial differentiation and unconstrained optimisation– whilst others- such as constrained optimisation - might be delivered as an additional advanced module. One possible way of delivering these concepts is set out below.

Section 2: Bi-Variate Functions

1. The concept of bi-variate functions

Students will already have come across such functions although they will not necessarily recognise the technical term for them. It is useful to simply re-enforce what students already know albeit in an unfamiliar language.

2. Presenting the concept of bi-variate functions

An effective way to deliver the concept is to start back at first principles and explain what bi-variate functions relating them to the many economic examples that students have already come across.

One could create a template for students to record exactly what the concept is, its different names, labels or synonyms or articulations– sometimes the simple language of mathematics can confuse students - and what its role is. A typical ‘template’ could look like this:

Definition of the concept		**Synonyms**
A bi-variate function is one in which one dependent variable is driven by two independent variables.	$Y = H(X_1, X_2)$	The dependent variable Y is a Function of two independent variables X_1 and X_2
Examples of application in economics and business Output (Y) is a function of labour (X_1) and capital (X_2) . Utility (U) is a function of the consumption of oranges (X_1) and apples(X_2). Government Tax Revenue (G) is a function of the tax rate (X_1) and income (X_2). **Particular Functional Forms for the Above** $Y = X_1^{\alpha} X_2^{\beta}$ $U = X_1 + \log(X_2)$ $G = X_1 X_2$		

3. Delivering the concept of bi-variate functions to small or larger groups

Larger groups could start by working through the template (see above) and moving onto the activities (see below). Much of the learning will be secured by students actually applying their knowledge and working through the problem sets.

Smaller groups could discuss the templates as a starting point and consider some of the discussion points. Small groups could be set a simple task where they have to research three 'real world' examples where a bi-variate function is used. They could prepare and deliver a 5 minute presentation to other small groups or to the whole larger group.

Links to the online question bank

Lecturers might wish to refer to the algebra questions on the online question bank posted at http://www.metalproject.co.uk/METAL/Resources/Question_bank/Algebra/index.html.

This includes some of the background material concerning algebraic functions and re-arranging equations.

Video clips

Lecturers could review the clips in Film Series 1 (see: http://www.metalproject.co.uk/Resources/Films/Mathematical_review/index.html) and Film Series 2 (See: http://www.metalproject.co.uk/Resources/Films/Linear_equations/index.html) which cover a wide range directly connected with an introduction to bi-variate analysis as well as more subsidiary material such as multiplier effects.

4. Discussion questions

Students could consider whether different labels for the variables change the nature or content of the function to reinforce the point that, for example, $Y = X_1^{\alpha} X_2^{\beta}$ has exactly the same content as $U = C_1^{\alpha} C_2^{\beta}$ but with different interpretations depending on what the labels mean.

5. Activities

ACTIVITY ONE

Learning Objectives

LO1. Students to consolidate basic meaning of bi-variate functions

LO2. Students to learn how to confidently use bi-variate functions in economics

Students are given the bivariate function : $Y = X_1^{\alpha} X_2^{\beta}$

They are put into small groups and asked to produce lists of all meaningful economic relationships connecting the variables utility, income, tax rate, consumption, disposable income, output, labour , capital, government revenue, interest rate, investment, apples, oranges, etc. which can be described by the function:

A complete answer involves specifying what the labels Y, $X_{1,}$ and X_2 mean and any restrictions on α and β are.

Task 1

Construct one example where α and β are both positive

Task 2

Construct one example where α is positive but β is negative

Task 3

Construct one example where α and β both take on specific numerical values.

ACTIVITY TWO

<u>Learning Objectives</u>

LO1. Students to consolidate basic meaning of bi-variate functions

LO2. Students to learn how to confidently use bi-variate functions

Task One

Students are given the bi-variate function : $Y = X_1^2 X_2^3$ and asked to complete the final column and comment on their ansers.

	X_1	X_2	Y
(i)	5	6	?
(ii)	5	7	?
(iii)	3	4	?
(iv)	3	5	?

ANSWERS

ACTIVITY ONE

There are a large number of solutions and, of course, the purpose is not to create a particular solution but rather to practise the confident use of bi-variate functions as used in economics.

Possible solutions include

Task 1: Y=utility, X_1 = oranges, X_2 =apples

Task 2: Y= Consumption, X_1 = income, X_2 =interest rate

Task 3: Y= Government revenue, X_1 = income, X_2 =tax rate, α and β =1

ACTIVITY TWO

Task One

5400

8575

576

1125

Comparing (i) with (ii) , note that Y increases by (8575 – 5400) = 3175 due to a unit increase in (X_2) whereas comparing (iii) with (iv) Y increases by only (1125-576)= 549 due to a unit increase in (X_2)

Message: The absolute change in Y, when only one independent variable changes, depends on the starting level of both independent variables.

6. Top Tips

Try and get students in small groups to work through the problem sets. Mixed ability groups would work best and help the most able to consolidate their understanding whilst also supporting weaker students. Consider using these sessions to create opportunities for students to **apply** their knowledge and understanding to real problems.

Students could prepare and deliver mini-presentations on bi-variate functions pertinent to their own lives, e.g. their own consumption functions.

7. Conclusion

Students will need to practise writing down bi-variate functions relevant to economics both in general form and in particular parameterised form too in order to feel confident but also so they can really understand what the uses of such functions are. Given the wide range of confidence and competence that lecturers are likely to see in their classes, it is also important to try and differentiate the material as much as possible: some students will need a quick recap recognising that they already have come across such functions whilst others will need more fundamental and comprehensive teaching.

Section 3: Partial Differentiation

1. The concept of partial differentiation

The label ‘partial differentiation’ is quite off-putting and students may initially struggle to understand what it actually means. The essential meaning, of course, is simply finding a marginal response or rate of change of the dependent variable as **one** independent variable changes with all other independent variables remaining constant.

2. Presenting the concept of partial differentiation

An effective way to deliver the concept is to explain how partial differentiation is related to ordinary differentiation and then relating them to the many economic examples that students have already come across. One could create a template for students to record exactly what the concept is, its different names, labels or synonyms or interpretations– sometimes the simple notation of mathematics can confuse students. A typical ‘template’ could look like this:

Definition of the concept		Interpretation
Partial differentiation is the process of calculating the derivative function of the dependent variable when only one independent variable is undergoing a small change but the other remains constant. For each bi-variate function there are therefore two partial derivative functions.	$H_1 = \delta H(X_1,X_2)/\delta X_1$ and $H_2 = \delta H(X_1,X_2)/\delta X_2$ are the two partial derivative functions.	H_1 measures the rate of change of the dependent variable Y when the independent variable X_1 changes but the independent variable X_2 is constant. It is the margin of Y with respect to X_1 holding X_2 constant. H_2 measures the rate of change of the dependent variable Y when the independent variable X_2 changes but the independent variable X_1 is constant. It is the margin of Y with respect to X_2 holding X_1 constant.

Examples of application in economics and business

If output (Y) is a function of labour (X_1) and capital (X_2) , say, $Y = H(X_1 ,X_2)$, then $H_1 (X_1 ,X_2)$ and $H_2 (X_1 ,X_2)$ respectively are the marginal product of labour function and the marginal product of capital function.

If Utility (U) is a function of the consumption of oranges (X_1) and apples(X_2), say $U = H(X_1 ,X_2)$, then $H_1 (X_1 ,X_2)$ and $H_2 (X_1 ,X_2)$ respectively are the marginal utility of oranges function and the marginal utility of apples function.

How to Calculate Partial Derivative Functions

Exactly the same as with ordinary derivatives. The rules of differentiation learnt in Guide 7 are unchanged. To find H_1, treat X_2 as a constant number so that effectively the bi-variate function is now just an ordinary function of X_1 and differentiate Y with respect to X_1. Exactly the same procedure operates when finding H_2 but we interchange the roles of X_1 and X_2 .

3. Delivering the concept of partial derivative functions to small or larger groups

The concept is best delivered through smaller groups who could discuss the templates as a starting point and consider some of the discussion points. Then individuals or teams could be set specific tasks and activities (see below). Much of the learning will be secured by students actually applying their knowledge and working through the problem sets.

Links to online question bank

There are 5 separate ‘sub-areas’ within the question bank centred on differentiation. These can be found at

http://www.metalproject.co.uk/METAL/Resources/Question_bank/Calculus/index.html

Video clips

Video clips for this material are located on the METAL website at

http://www.metalproject.co.uk/Resources/Films/Differential_equations/index.html

4. Discussion questions

Why do we refer to partial derivatives as functions of X_1 **and** X_2?

This question could help to reinforce the point that the specific numerical value of H_1 depends not only on the starting value of X_1 (from which we take a small change) but also on the constant value of X_2.

5. Activities

ACTIVITY ONE

<u>Learning Objectives</u>

LO1. Students to consolidate basic meaning of partial derivative functions

LO2. Students to learn how to confidently calculate partial derivative functions in economics

Students are given the bivariate function : $Y = X_1^2 X_2^3$ which reduces to the univariate function $Y=8 X_1^2$ when $X_2= 2$.

Students are divided into two groups. Students in Group A find the ordinary derivative of $Y=8 X_1^2$ (answer is 16 X_1) and evaluate it for X_1 = 2, 3, 4 etc. Students in Group B find H_1 , the partial derivative of $Y= X_1^2 X_2^3$ w.r.t. X_1 (answer is $2X_1 X_2^3$) and evaluate it for (X_1,X_2) = (2,2), (3,2), (4,2) etc

They then compare notes by reporting their answers and filling up the following table:

Value of X_1	**Group A** **Ordinary derivative of $Y=8 X_1^2$**	**Group B** **H_1 ,Partial derivative of $Y= X_1^2 X_2^3$ w.r.t. X_1 (evaluated when X_2=2)**
2		
3		
4		

The numerical results in Column 2 and 3 should be the same.

Task 1

Create a table similar to the above when X_1 is constant at 3 but X_2 is varying.

Task 2

Draw a rough sketch of H_1against X_1 holding X_2 constant at X_2 =2

Task 3

Draw a rough sketch of H_1against X_2 holding X_1 constant at X_1 =4

ACTIVITY TWO

Learning Objectives

LO1. Students to consolidate the idea that the partial derivative (like the ordinary derivative) is a function (not a number)

Task One

Students are given the bi-variate function : $Y = H(X_1, X_2) = X_1^2 X_2^3$ and asked to complete the following table and comment on their answers

	X_1	X_2	H_1	H_2
(i)	5	6	**?**	**?**
(ii)	5	7	**?**	**?**
(iii)	3	4	**?**	**?**
(iv)	3	5	**?**	**?**

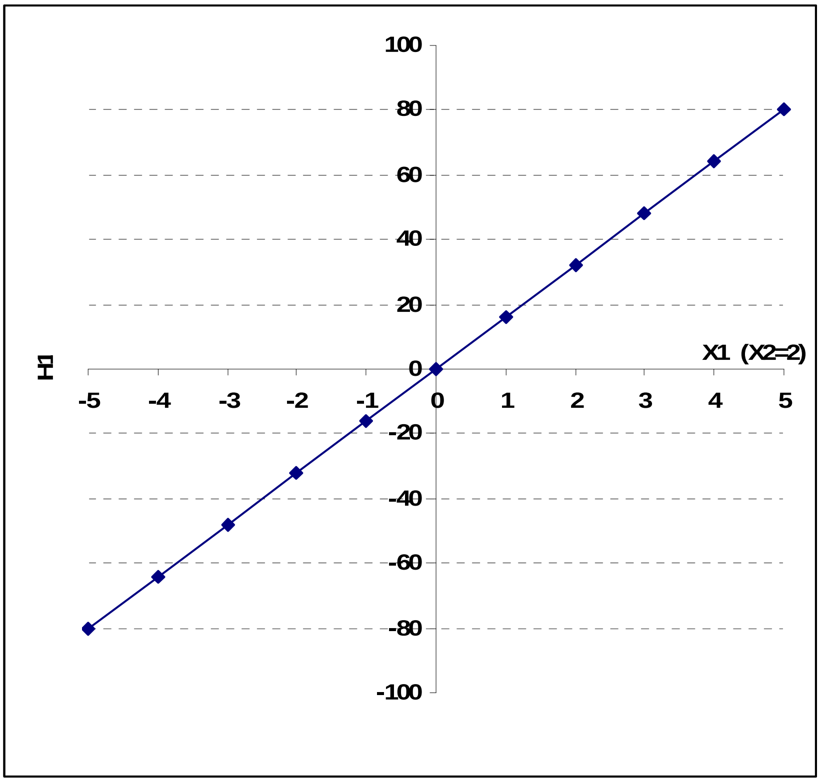
ANSWERS

ACTIVITY ONE

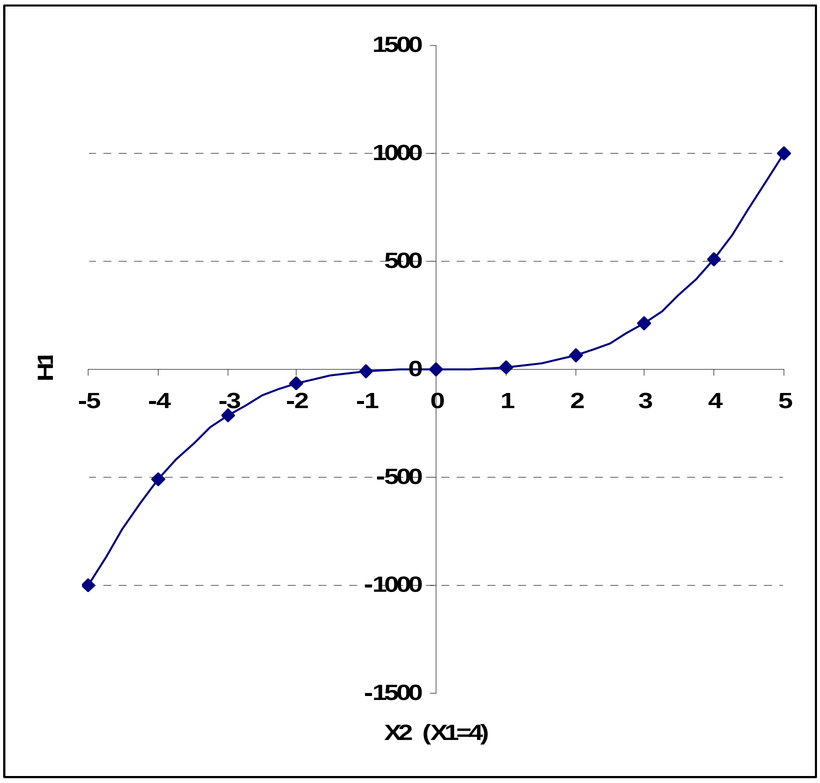
Task One

In the tables, columns 2 and 3 should produce the same answers. Students should appreciate why they are equivalent. Group A is reducing the bi-variate function to a univariate function first and then differentiating whilst Group B is differentiating first and then setting a specific value for the constant X_2.

Task Two



Task Three



ACTIVITY TWO

Task One

2160, 2700

3430, 3234

576, 432

750, 675

Comparing (i) with (ii) and (iii) with (iv) note that **both** H_1 and H_2 numerically increase due to a unit increase in (X_2)

Message: The value of both partial derivatives (as only one independent variable changes) depends on the starting level of both independent variables.

6. Top Tips

Consider using these sessions to create opportunities for students to mechanically calculate partial derivative functions and then compute the numerical value of these functions for different values of the starting point (X_1, X_2)

7. Conclusion

Students will need to practise writing down partial derivative functions relevant to economics (marginal productivity of labour and capital from a production function, marginal utlity of both goods from a utility function etc.) both in general form and in particular parameterised form too in order to feel confident but also so they can really understand what the uses of such functions are.

Section 4: Unconstrained Optimisation

1. The concept of unconstrained optimisation

Students need to be clear that optimisation in economics simply refers to the actions by which individual economic agents do as well as they can. Thus consumers who seek to maximise utility are optimising when they choose consumption bundles which yield the highest level of utility. Similarly firms who seek to maximise profits given production functions, prices of out put and inputs, are optimising when they choose input levels which yield the highest level of profits. In this section we present only the class of unconstrained optimisation problems in which there are no restrictions placed on permissible values of (X_1, X_2) in carrying out the optimisation. Constrained optimisation problems, in which, for example, the firm cannot hire more than a certain amount of capital or labour, are the subject of the next section.

2. Presenting the concept of unconstrained optimisation

An obvious way to present the concept is by reference to economic examples. Care should be taken here as most examples from economics involve constrained optimisation. However, the theory of the firm presents us with an opportunity. Consider a firm which seeks to maximise profits. Given its production function $Q= F(X_1, X_2)$ where X_1 = labour input, and X_2 = capital input and P is the price of output, and R_1 and R_2 the prices per unit of labour and capital respectively, the firm's profit (H) can be written as:

$$H(X_1, X_2) = P\,F(X_1, X_2) - R_1 X_1 - R_2 X_2$$

If the firm seeks to maximise profits, then it must choose (X_1, X_2) to make the consequential value of H (above) as large as possible. This is optimising behaviour- doing the best one can, given ones objectives. As there are no restrictions on the choice of (X_1, X_2), this is an unconstrained optimisation problem. Other examples from economics come from the multi-product firm and the discriminating monopolist.

3. Delivering the concept and mathematics of unconstrained optimisation

The obvious analogy with single variable Optimisation (Guide 7) can be usefully exploited here. Students need to be reminded of the two part solution used for uni-variate optimisation. First, find the value of the independent variable that ensures you are either on top of a hill or at the

bottom of a valley (the first order conditions). Secondly, ensure that the function you are maximising has the appropriate curvature (the second Order conditions).
The first order conditions (FOC) are analogues of the uni-variate case. i.e:

The FOC for maximising $Y = H(X_1,X_2)$ are:

$H_1 (X_1 ,X_2) = 0$

$H_2 (X_1 ,X_2) = 0$

This pair of first order conditions is in essence a pair of simultaneous equations in (X_1 ,X_2). In most cases, they will not be linear and hence some flexibility and mathematical "tricks" will be required to solve them.

The Second Order conditions (SOC) involve higher order differentiation, just as in the Univariate case. However with bi-variate functions, each first partial derivative function has two partial derivatives. Since H_1 is a function of both X_1 and X_2, we can differentiate H_1 with respect to both X_1 and X_2 (one at a time). Differentiating H_1 w.r.t to X_1(holding X_2 constant), we obtain $\delta H_1 (X_1 ,X_2)/ \delta X_1$ denoted conveniently as H_{11} and differentiating H_1 w.r.t to X_2(holding X_1 constant), we obtain $\delta H_1 (X_1 ,X_2)/ \delta X_2$ denoted conveniently as H_{12}. Similar treatment of $H_2 (X_1 ,X_2)$ yields H_{22} and H_{21} respectively.

The SOC for maximising $Y = H(X_1,X_2)$ then are:

$\Delta = [H_{11} H_{22} - H_{12} H_{21}] > 0$ **and** $H_{11} < 0$

Further simplification arises from the fact that for most "regular" functions, $H_{12} = H_{21}$.

Links to online question bank

There are 5 separate 'sub-areas' within the question bank centred on differentiation. These can be found at:

http://www.metalproject.co.uk/METAL/Resources/Question_bank/Calculus/index.html

Questions regarding optimisation are located at:

http://www.metalproject.co.uk/METAL/Resources/Question_bank/Economics%20applications/index.html

Video clips

Video clips for this material are located on the METAL website at:

http://www.metalproject.co.uk/Resources/Films/Differential_equations/index.html

4. Discussion questions

How do we interpret the FOC in terms of marginal benefit and marginal cost?
How do we handle minimisation problems? The reason for not discussing minimisation techniques separately is that every minimisation problem can first be converted into a maximisation problem by inserting a negative sign in front of the objective function and then proceeding to maximise as above. In other words if we are required to Minimise J (X_1, X_2), we simply solve the problem:

Maximise $H(X_1, X_2) = - J(X_1, X_2)$

There is no need to clutter students with thinking they need a special technique for handling minimisation.

5. Activities

ACTIVITY ONE

Learning Objectives

LO1. Students to consolidate writing down an objective function which is to be maximised.

LO2. Students to learn how to obtain the FOC

LO3. Students learn how to solve the FOC

Task One

Students are given the following problem. A multiproduct firm has cost function given by $TC=ax^2 +bxy +cy^2 + K$ where x and y are the quantities produced of its two outputs. The prices of these are fixed in world markets at p and q respectively. How much of each output should the firm produce?

Students should now be split into four groups and each group be given the following tasks. The groups should not be in contact, i.e. each group works in isolation from all others.

Group A: Write down the optimisation problem in standard form, making the notation clear.

Group B: Given the function $\pi = px + qy - [ax^2 + bxy + cy^2 + K]$, find the partial derivative functions π_x and π_y.

Group C: Given the problem and the partial derivatives $\pi_x = p - 2ax - by$ and $\pi_y = q - bx - 2cy$

What are the FOC?

Group D: Find the solution to the equations $p - 2ax - by = 0$ and $q - bx - 2cy = 0$

ACTIVITY TWO

LO1. Students learn how to operate SOC to check for a maximum

Task One:

Continuing with the above problem students are now divided again into three groups. Each group works in isolation as before and is assigned the following tasks.

Group A: Calculate the second partial derivative functions: π_{xx}, π_{xy} and π_{yy} ,π_{yx}

Group B: Given the answers to the second partial calculations, find the value of Δ and π_{xx}

Group C: Using economic intuition and the fact that $\Delta = 4ac - b^2$ and $\pi_{xx} = -2a$, discuss whether the SOC are satisfied.

ANSWERS

ACTIVITY ONE

Task One

Group A : Choose x and y to Maximise $\pi = px + qy - [ax^2 + bxy + cy^2 + K]$

Group B: $\pi_x = p - 2ax - by$ and $\pi_y = q - bx - 2cy$

Group C: $p - 2ax - by = 0$ and $q - bx - 2cy = 0$

Group D: $x^* = [2cp-bq]/[4ac - b^2]$ and $y^* = [2aq - bp]/[4ac - b^2]$

Then the sequence of answers is put together by the Lecturer and the whole group discusses the question “ Are the values of x* and y* found by this procedure the firm’s optimal output levels?” [Answer: We cannot be sure because the Second Order Conditions have not been checked] **.**

ACTIVITY TWO

Task One

Group A: -2a, -b, -2c, - b. Note that $\pi_{xy} = \pi_{yx}$, both being equal to – b.

Group B: Δ = 4ac – b^2 and π_{xx} = -2a

Group C : It is immediately obvious that part of the SOC is satisfied because π_{xx} is clearly negative. But what about Δ? Economic intuition tells us that if the price of y, i.e. q goes up, it would be utterly perverse for the firm to sell less y. Similarly for x. In other words, we expect supply curves to be upward sloping!

From the solutions to FOC we note that x* = [2cp-bq]/[4ac –b^2] and y*= [2aq –bp]/[4ac –b^2]. The impact of p on x* and of q on y* will be positive only if 2c/[4ac –b^2] and 2a/[4ac –b^2] is strictly positive. But 2c and 2a are clearly positive. Therefore in order for upward sloping supply curves, we require [4ac –b^2] to be strictly positive. But [4ac –b^2] is precisely the value of Δ. Hence, Δ > 0 and the SOC are satisfied.

6. Top Tips

Be sure to get students to work through numerical exercises following the sequential procedure above. It is easy to replace the parameters a,b,c in the illustrative problem above with numerical values. The more able students might be given a problem on finding input demand functions for a Cobb-Douglas production function; they will have to be confident of solving simultaneous equations which are only linear once log transformations have been used.

Ask students to think about the role of K, in particular, how large must K get before the firm shuts down? Finally, it may help to give students a template for Unconstrained Optimisation. A typical ‘template’ could look like this:

Definition of the concept	Method	Interpretation
An Unconstrained Optimisation problem is one in which one seeks to maximise (or minimise) some objective function of two(or more) independent variables without any restrictions on what values these independent variables can take.	**Use the underlying economics to write the problem in the form:** **Max $Y = H(X_1, X_2)$** **Obtain all relevant partial derivative functions, i.e. H_1, H_2, H_{11}, H_{22} H_{12}, and H_{21}** **2 Obtain FOC:** **$H_1(X_1, X_2) = 0$ and $H_2(X_1, X_2) = 0$** **3.Solve the above pair of equations to obtain (X_1^*, X_2^*)** **Compute Δ and H_{11} and hence check SOC**	The FOC will always carry the interpretation that $(X_1, X_2)^*$ the optimal levels of the activities (X_1, X_2) will be when the Marginal Benefit equals the Marginal Cost for both activities simultaneously. The interpretation of the typical solutions one obtains depends on the context, but often they will represent optimal response functions (supply curves, input demand curves, etc.)

7. Conclusion

Unconstrained optimisation is of some value in itself. However, mastering the techniques of unconstrained optimisation is an absolute pre-requisite to understanding Constrained Optimisation which is the bed rock on which virtually all conventional microeconomics is founded.

Section 5: Constrained Optimisation

1.Understanding the concept of constrained optimisation

A large class of economic problems consist of characterising an economic agent (firm, consumer, government, trade union etc) seeking to **maximise** some objective (utility, profits, welfare, rent) subject to some constraints (budget sets, production functions, resource availability, labour demand).

The common mathematical method for handling all these problems is the technique of **Constrained Optimisation**. The key to understanding Constrained Optimisation is understanding the concept of constraints and their role. Constraints place a restriction on the freedom of choice of the optimising agent. In an unconstrained optimisation problem, the agent can choose *any* values of the independent variables. By contrast, in a constrained problem, the agent can only choose from amongst a restricted well defined set. This set is called the **Feasible Set** (denoted by F) and is the main new concept in this area.

2. Presenting the concept of unconstrained optimisation

The basic ingredients of the problem are (i) the agent's **objective function** $H(X_1,X_2)$ and one or more **constraints** of the form $G(X_1 ,X_2) \leq c$, where the pair (X_1,X_2) are the **choice variables** and c is a parameter. The function $H(X_1,X_2)$ shows how the agents choices (X_1,X_2) affect his objective and the constraint $G(X_1 ,X_2) \leq c$ shows the **Feasible Set** from which the agent must make his choice of (X_1,X_2). Extensions to more than two variables adds no extra complications.

The simplest way to present the concept is by reference to a well known example taken from standard micro economics. For example, the well known consumer problem can be stated as:

"A consumer has Utility Function $U(X_1,X_2)$ and faces fixed prices of p_1 and p_2 and has a fixed income of M. What is his optimum consumption bundle? ". This example is used to illustrate the constrained optimisation technique

3. Delivering the concept and mathematics of unconstrained optimisation

Formally, we can write the typical problem as:

Choose (X_1,X_2) so as to Max $H(X_1,X_2)$ s.t. $G(X_1,X_2) \le c$

The diagrammatic method students are familiar with usually consists of drawing **level contours** of the objective function and looking for the point of tangency between some level contour and the boundary of the feasible set. Such "tangency solutions" are then usually described in terms of "marginal" conditions. In what follows, we will examine the methodology underlying such "tangency solutions". In particular students will need to investigate (i) the circumstances under which the "tangency solutions" do actually characterise a maximum, and (ii) a method for locating the "tangency solutions" mathematically. We illustrate this method using the consumer problem as a learning tool.

Worked Example

A consumer has Utility Function $U(X_1,X_2)$ and faces fixed prices of p_1 and p_2 and has a fixed income of M. What is his optimum consumption bundle?

Let us rewrite the problem formally as:

Max $U(X_1,X_2) = H(X_1,X_2)$ s.t. $p_1X_1 + p_2X_2 = G(X_1,X_2) \le M = c$

Note that in this problem the function G( ) takes on a linear form which is arising from the underlying economics of the problem (prices and income are fixed). The Feasible set - denoted by F – consists of all non-negative (X_1,X_2) which satisfy the linear budget constraint $p_1X_1 + p_2X_2 \le M$. Since the budget line intercepts on each of the axes are (M/p_1) and (M/p_2) respectively , the Feasible Set F consists of the triangle O , (M/p_1) and (M/p_2) as in the familiar diagram.

We normally describe the solution by imposing the set of indifference curves corresponding to $U(X_1,X_2)$ onto the feasible set and looking for the unique point where some Indifference Curve is tangent to the linear boundary of F. This "tangency solution" depicts the consumers' optimal choices. It implies for instance that slope of IC = slope of budget line or MRS = (p_1/p_2).

But these "tangency solutions" are only first order conditions (FOC). They are necessary but not sufficient. We need to be careful about the sufficient conditions which will indeed ensure that solving the FOC does locate the maximum. Exploring the sufficient conditions requires an understanding of convexity.

Convexity and Optimisation.

Students will need to note that a set which consists of a collection of points is said to be a Convex Set if and only if every point on the straight line joining any two points that belong to the set also belongs to the set. Examples of convex sets are the areas enclosed by a circle or by a square. By contrast consider the area outwith a circle (but including the circumference of the circle). This set would be a non-convex set.

Students will need to be aware that convex sets play a major role in the theory of optimisation. Consider the set of points $H(X_1,X_2) \geq h$, i.e. the set of points which is no worse than the level contour value of h. We refer to this set as the "Superior" set S. Every member of S gives a value of H which no less than h. S can be either convex or non-convex.

Similarly consider the feasible set F given by $G(X_1, X_2) \leq c$ and $(X_1,X_2) \geq 0$. This set can also be convex or non-convex.

Separation Theorem:

If both S and F are convex sets, then there exists a hyper-plane (HP) which separates S and F in the sense that: (i) for a suitable choice of h, the sets S and F have only one common point and (ii) the separating HP passes through this point.

Lecturers will want to highlight the fact that one implication of the Separation Theorem is that no point belonging to S lies below HP and no point belonging to F lies above HP.

Optimisation and Separation

A corollary of the above is that if F and S are both convex so that the Separation Theorem holds, then the FOC (tangency solutions) do in fact locate a maximum for the problem :

Max $H(X_1,X_2)$ s.t. $G(X_1, X_2) \leq c$ and $(X_1,X_2) \geq 0$.

It further turns out the separating hyper-plane HP is of great significance to economists.

The final step in the Optimisation technique consists of a method which allows us to write down the FOC and solve them to obtain the solution (X_1^*,X_2^*)

Lagrange's Method:

Students will need to have good grasp of the Lagrangean method so they can tackle more complex examples (and since using the graphical method and appealing to tangency solutions to establish the FOC is not always helpful).

The general problem students will need to solve is:

$$\text{Max } H(X_1,X_2) \text{ s.t. } G(X_1,X_2) \le c$$

And this can written down using the Lagrangean:

$$L = H(X_1,X_2) - \lambda\,[G(X_1,X_2) - c]$$

Point out to students that what the Lagrangean does is simply subtract from the objective H a penalty for violating the constraint. This is a way of ensuring that the constraint is not broken. The "price" paid for breaking the constraint is in fact λ, a new non-negative variable that is introduced to help solve the original problem by converting the original problem into an unconstrained problem. If we now maximise L (X_1,X_2) the first order conditions are:

$$L_1 = H_1 - \lambda G_1 = 0$$

$$L_2 = H_2 - \lambda G_2 = 0$$

But these two equations cannot solve for three unknowns viz. (X_1,X_2) and λ. One more equation is required. This comes from the constraint. If the solution to the problem involves a choice of (X_1,X_2) such that the constraint is not binding, then $G(X_1,X_2) < c$ in which case λ must equal 0 (there is no penalty for a small increase in (X_1,X_2)). However, if the constraint is binding , then $G(X_1,X_2) = c$ and $\lambda>0$ (a penalty must be paid if (X_1,X_2) is increased). This idea is summed in the last part of the FOC as:

$$\lambda\,[G(X_1,X_2) - c] = 0$$

Combining these results we get the fundamental **Theorem of Constrained Optimisation** which states:

To solve the problem Max $H(X_1,X_2)$ s.t. $G(X_1, X_2) \leq c$,

Define:

the associated Lagrangean L given by: $L = H(X_1,X_2) - \lambda [G(X_1, X_2) - c]$.

the "superior" set S given by all (X_1,X_2) for which $H(X_1,X_2) \geq h$, for any h, and

the feasible set F given by all (X_1,X_2) for which $G(X_1, X_2) \leq c$ and $(X_1,X_2) \geq 0$

Then the necessary FOC are:

$L_1 = H_1 - \lambda G_1 = 0$

$L_2 = H_2 - \lambda G_2 = 0$

$\lambda [G(X_1, X_2) - c] = 0$

If the sets S and F are convex, then the solution to the FOC viz. $(X_1^*, X_2^*, \lambda^*)$ are the solutions to the problem. There are formal methods one can use to check for the convexity of S and F. But for many purposes a simple diagrammatic check will suffice.

Links to online question bank

There are 5 separate 'sub-areas' within the question bank centred on differentiation. These can be found at:

http://www.metalproject.co.uk/METAL/Resources/Question_bank/Calculus/index.html

Questions regarding optimisation are located at:

http://www.metalproject.co.uk/METAL/Resources/Question_bank/Economics%20applications/index.html

Video clips

Video clips for this material are located on the METAL website at:

http://www.metalproject.co.uk/Resources/Films/Differential_equations/index.html

4. Discussion questions

a) How do we interpret the FOC?

The FOC (i) and (ii) have the standard marginal benefit = marginal cost interpretation. However, the third FOC viz.

$\lambda\ [G(X_1, X_2) - c] = 0$ is unusual.

What it says is that either the constraint is binding so that $G(X_1, X_2) = c$, in which case λ the penalty for increasing (X_1, X_2) even a little bit is strictly positive, or, the constraint is slack so that $G(X_1, X_2) < c$, in which case λ the penalty for increasing (X_1, X_2) a little bit is zero. Students should try and develop the intuition underlying these.

b) How do we handle minimisation problems?

Once again, we resort to the same trick used in handling unconstrained optimisation. No new technology is required.

c) What information is contained in the value of λ?*

This is a rather important and frequently neglected area. The value of λ^* measures the impact on the maximised value of the objective of an increase in c. Formally, if $(X_1^*, X_2^*, \lambda^*)$ are the solutions to the problem, so that the maximised value of the objective is $H^*(X_1^*, X_2^*)$, then

$$dH^*/dc = \lambda^*$$

Note what this is saying. dH^*/dc measures the impact on the maximised value of the objective of an increase in c. If the constraint is slack ($\lambda^*=0$), then increasing the choice set by increasing c will not make the agent any better off. Hence dH^*/dc will equal zero.

5. Activities

ACTIVITY ONE

<u>Learning Objectives</u>

LO1: Students to learn how to obtain the FOC

LO2: Students learn how to solve the FOC

LO3: Students learn how to interpret the FOC

Task One

Students are given the following problem:

(a) Maximise $H = p_1X_1 + p_2X_2$ s.t . $G = X_1^2 + X_2^2 \leq r^2$ and $(X_1, X_2) \geq 0$

(b)Using your answer to (a) write down the FOC and interpret

(c) Using your answer to (c) solve these three equations

ACTIVITY TWO

Learning Objectives

LO1. Students consolidate solving and interpretation of FOC.

This is an individual activity. All students are given the following problem:

A small country is seeking to maximise its export revenue by selling its coffee and coconuts on the world market where they fetch $ 6 and $ 8 per ton respectively. In producing coffee and coconuts, the country is constrained by its production set which is given by:

$$X_1^2 + X_2^2 \leq 100$$

Where X_1 denotes coffee production in tons and X_2 denotes coconut production in tons.

Task One

Find the optimum production values X_1^* and X_2^* of coffee and coconuts and the export revenue earned. At what rate would the export revenue increase if the production set constraint was slightly relaxed.

Task Two

Re-solve the problem if the production constraint was changed to $X_1^2 + X_2^2 \leq 121$. By how much does export revenue change? How could you obtain this answer without resolving the whole problem? If the change in the production constraint to be shifted to $X_1^2 + X_2^2 \leq 121$ is achievable by buying a technology, what is the most the country should pay for this technology?

ACTIVITY THREE

Learning Objectives

LO1. Students learn how to operate convexity conditions to check for a maximum

Task One

Students are asked to draw shapes of convex and non convex sets

Task Two

Students are asked to note common economic problems in which the relevant constraints do indeed form a convex set.

Task Three

Students are asked to note how to identify convex superior sets that arise commonly in economics.

Task Four

Students should check whether the convexity conditions are satisfied for the problem above.

ANSWERS

ACTIVITY ONE

Task One

(a) Answer: $L = p_1X_1 + p_2X_2 - \lambda [X_1^2 + X_2^2 - r^2]$

It is very important that students write the Lagrangean correctly, particularly noting the NEGATIVE sign between the objective and the constraint. They also need to note that in this problem r^2 is c.

(b) Answer:

$L_1 = p_1 - 2 \lambda X_1 = 0$ (1)

$L_2 = p_2 - 2 \lambda X_2 = 0$ (2)

$\lambda [X_1^2 + X_2^2 - r^2] = 0$

(c) Answer:

To solve these three equations we start with 1) and 2).

Rewrite (1) and (2) as:

$p_1 = 2\ \lambda\ X_1$ (4)

$p_2 = 2\ \lambda\ X_2$ (5)

Divide 4) by 5) to get:

$p_1 / p_2 = X_1 / X_2$ (6)

From 4) and 5) since both p_1 and p_2 are strictly positive, it follows that (X_1^*, X_2^*) and λ^* are strictly positive.

Since $\lambda^* > 0$, it follows that:

$X_1^2 + X_2^2 - r^2 = 0$. (7)

Now solve 6 and 7 to get :

$X_1^* = r\ p_1/\sqrt{[p_1^2 + p_2^2]}$ and $X_2^* = r\ p_2/\sqrt{[p_1^2 + p_2^2]}$

To find the value of λ^*, substitute $X_1^* = r\ \sqrt{p_1^2/[p_1^2 + p_2^2]}$ in 4) to get:

$\lambda^* = \{\sqrt{[p_1^2 + p_2^2]}\}/2r$

Thus the full solution is:

$X_1^* = r\ p_1/\sqrt{[p_1^2 + p_2^2]}$ and $X_2^* = r\ p_2/\sqrt{[p_1^2 + p_2^2]}$ and $\lambda^* = \{\sqrt{[p_1^2 + p_2^2]}\}/2r$

As a final step substitute (X_1^*, X_2^*) from above into H to obtain:

$H^* = r\, p_1^2/\sqrt{[p_1^2 + p_2^2]} + r\, p_2^2/\sqrt{[p_1^2 + p_2^2]}$

ACTIVITY TWO

TASK ONE

This is just a numerical version of the problem analysed in Activity 1.

$L=6X_1 + 8X_2 – \lambda [X_1^2 + X_2^2 – 100]$

FOC are:

(1): $L_1 = 6 - 2 \lambda X_1 = 0$

(2): $L_2 = 8 - 2 \lambda X_2 = 0$

(3): $\lambda[X_1^2 + X_2^2 – 100] = 0$

From (1) and (2), ¾= X_1/ X_2 (A)

From (A) both X_1 and X_2 are strictly positive. Hence from (1) so is $\lambda > 0$ hence from (3)

$X_1^2 + X_2^2 – 100 = 0$ (B)

Substitute (A) into (B) to get:

($X_1^* = 6$, $X_2^* = 8$) and $R^* = 100$. Substitute $X_1^* = 6$ into (1) to get $\lambda^* = ½$.

The rate at which the export revenue increase if the production set constraint was slightly relaxed = $\lambda^* = ½$. $R^* = 100$

TASK TWO

If the constraint changes to $X_1^2 + X_2^2 \leq 121$, then one can re-solve the whole problem to get:

($X_1^{**} = 6.6$, $X_2^{**} = 8.8$) and $R^{**} = 110$. Hence $dR^* = 110-100= 10$.

Much easier to solve by using the information contained in the Lagrange Multiplier, i.e. $dR^*/d(r^2)= \lambda^*$

Since $\lambda^* = ½$, and $d(r^2)=121-100=21$, then $dR^* = ½ \times 21 = 10.5$ – which is approximately same as by long method.

ACTIVITY THREE

Task One

Convex sets are triangles, circles, semi circles, quarter circles, rectangles squares etc. The best example of a non-convex set is a doughnut.

Task Two

Consumer theory (budget set is a triangle, hence convex), Producer theory (production function with non-increasing returns to scale)

Task Three

Indifference curve shapes or iso-profit lines which are linear.

Task Four

Yes they are since $H= p_1X_1 + p_2X_2$ is linear and hence the "Superior" set S [H(,) ≥ h] for any h is a convex set. Furthermore since the Feasible set F is a quarter circle and hence convex.

6. Top Tips

It is vital that students build confidence slowly. At first, one should get them to mechanically solve FOC, then at the next stage interpret, and finally check for the convexity conditions. It is not helpful to put together the whole problem until these three stages have each been separately mastered.

7. Conclusion

This is quite a challenging area. However, it is the very heart of conventional microeconomics. Sometimes students may well not immediately appreciate the logic behind these techniques or the interpretive value to economists. However, with practice, confidence at the purely mechanical aspects will improve. Once this stage is reached, the deeper understanding and interpretive power of the Lagrangean method will eventually follow.