Bi-Variate Functions - ACTIVITES

ACTIVITY ONE

Learning Objectives

LO1. Students to consolidate basic meaning of bi-variate functions

LO2. Students to learn how to confidently use bi-variate functions in economics

Students are given the bivariate function : $Y = X_1^{\alpha} X_2^{\beta}$

They are put into small groups and asked to produce lists of all meaningful economic relationships connecting the variables utility, income, tax rate, consumption, disposable income, output, labour , capital, government revenue, interest rate, investment, apples, oranges, etc. which can be described by the function:

A complete answer involves specifying what the labels Y, X_1, and X_2 mean and any restrictions on α and β are.

Task 1

Construct one example where α and β are both positive

Task 2

Construct one example where α is positive but β is negative

Task 3

Construct one example where α and β both take on specific numerical values.

ACTIVITY TWO

Learning Objectives

LO1. Students to consolidate basic meaning of bi-variate functions

LO2. Students to learn how to confidently use bi-variate functions

Task One

Students are given the bi-variate function : $Y = X_1^2 X_2^3$ and asked to complete the final column and comment on their ansers.

	X_1	X_2	Y
(i)	5	6	?
(ii)	5	7	?
(iii)	3	4	?
(iv)	3	5	?

Bi-Variate Functions - ANSWERS

ACTIVITY ONE

There are a large number of solutions and, of course, the purpose is not to create a particular solution but rather to practise the confident use of bi-variate functions as used in economics.

Possible solutions include

Task 1: Y=utility, X_1 = oranges, X_2 =apples

Task 2: Y= Consumption, X_1 = income, X_2 =interest rate

Task 3: Y= Government revenue, X_1 = income, X_2 =tax rate, α and β =1

ACTIVITY TWO

Task One

5400

8575

576

1125

Comparing (i) with (ii) , note that Y increases by (8575 – 5400) = 3175 due to a unit increase in (X_2) whereas comparing (iii) with (iv) Y increases by only (1125-576)= 549 due to a unit increase in (X_2)

Message: The absolute change in Y, when only one independent variable changes, depends on the starting level of both independent variables.

Partial Differentiation - ACTIVITIES

ACTIVITY ONE

Learning Objectives

LO1. Students to consolidate basic meaning of partial derivative functions

LO2. Students to learn how to confidently calculate partial derivative functions in economics

Students are given the bivariate function : $Y = X_1^2 X_2^3$ which reduces to the univariate function $Y=8 X_1^2$ when $X_2= 2$.

Students are divided into two groups. Students in Group A find the ordinary derivative of $Y=8 X_1^2$ (answer is $16 X_1$) and evaluate it for $X_1 = 2, 3, 4$ etc. Students in Group B find H_1 , the partial derivative of $Y= X_1^2 X_2^3$ w.r.t X_1 (answer is $2X_1 X_2^3$) and evaluate it for $(X_1,X_2) = (2,2), (3,2), (4,2)$ etc

They then compare notes by reporting their answers and filling up the following table:

Value of X_1	**Group A** **Ordinary derivative of $Y=8 X_1^2$**	**Group B** **H_1 ,Partial derivative of $Y= X_1^2 X_2^3$ w.r.t. X_1 (evaluated when X_2=2)**
2		
3		
4		

The numerical results in Column 2 and 3 should be the same.

Task 1

Create a table similar to the above when X_1 is constant at 3 but X_2 is varying.

Task 2

Draw a rough sketch of H_1against X_1 holding X_2 constant at $X_2 =2$

Task 3

Draw a rough sketch of H_1against X_2 holding X_1 constant at $X_1 =4$

ACTIVITY TWO

Learning Objectives

LO1. Students to consolidate the idea that the partial derivative (like the ordinary derivative) is a function (not a number)

Task One

Students are given the bi-variate function: $Y = H(X_1, X_2) = X_1^2 X_2^3$ and asked to complete the following table and comment on their answers

	X_1	X_2	H_1	H_2
(i)	5	6	**?**	**?**
(ii)	5	7	**?**	**?**
(iii)	3	4	**?**	**?**
(iv)	3	5	**?**	**?**

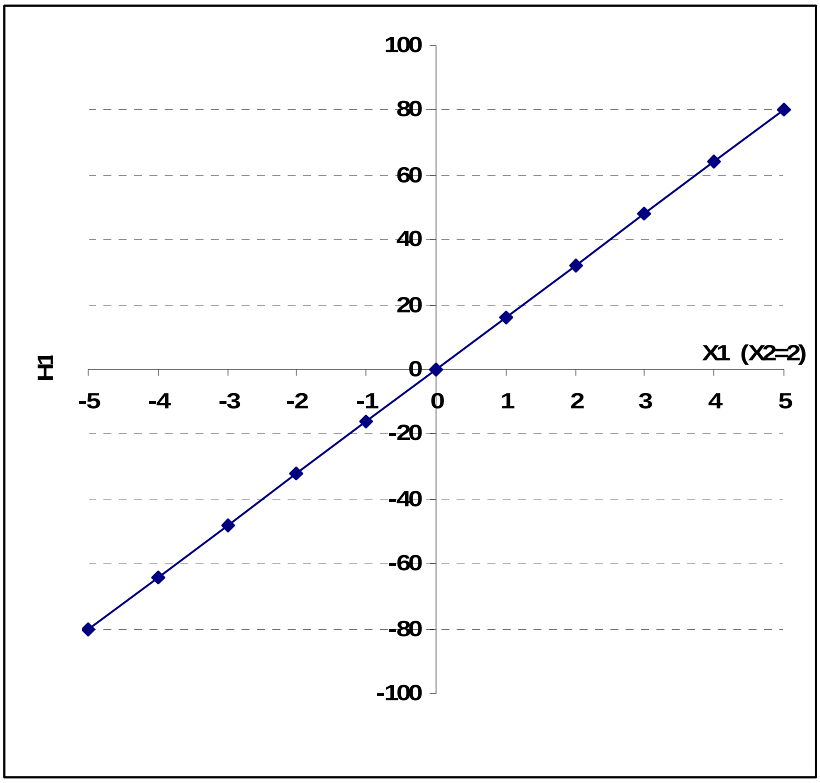
Partial Differentiation - ANSWERS

ACTIVITY ONE

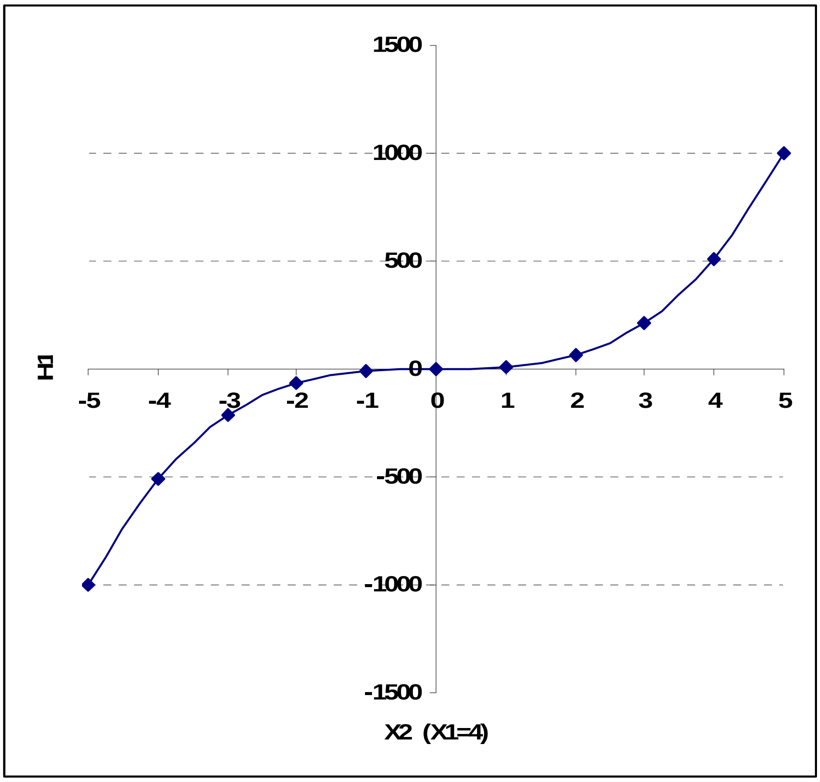
Task One

In the tables, columns 2 and 3 should produce the same answers. Students should appreciate why they are equivalent. Group A is reducing the bi-variate function to a univariate function first and then differentiating whilst Group B is differentiating first and then setting a specific value for the constant X_2.

Task Two



Task Three



ACTIVITY TWO

Task One

2160, 2700

3430, 3234

576, 432

750, 675

Comparing (i) with (ii) and (iii) with (iv) note that **both** H_1 and H_2 numerically increase due to a unit increase in (X_2)

Message: The value of both partial derivatives (as only one independent variable changes) depends on the starting level of both independent variables.

Unconstrained Optimisation - ACTIVITIES

ACTIVITY ONE

Learning Objectives

LO1. Students to consolidate writing down an objective function which is to be maximised.

LO2. Students to learn how to obtain the FOC

LO3. Students learn how to solve the FOC

Task One

Students are given the following problem. A multiproduct firm has cost function given by $TC=ax^2 +bxy +cy^2 + K$ where x and y are the quantities produced of its two outputs. The prices of these are fixed in world markets at p and q respectively. How much of each output should the firm produce?

Students should now be split into four groups and each group be given the following tasks. The groups should not be in contact, i.e. each group works in isolation from all others.

Group A: Write down the optimisation problem in standard form, making the notation clear.

Group B: Given the function $\pi = px + qy – [ax^2 +bxy +cy^2 + K]$, find the partial derivative functions π_x and π_y.

Group C: Given the problem and the partial derivatives $\pi_x = p – 2ax –by$ and $\pi_y = q – bx – 2cy$

What are the FOC?

Group D: Find the solution to the equations $p – 2ax –by = 0$ and $q – bx – 2cy = 0$

ACTIVITY TWO

LO1. Students learn how to operate SOC to check for a maximum

Task One:

Continuing with the above problem students are now divided again into three groups. Each group works in isolation as before and is assigned the following tasks.

Group A: Calculate the second partial derivative functions: π_{xx}, π_{xy} and π_{yy} ,π_{yx}

Group B: Given the answers to the second partial calculations, find the value of Δ and π_{xx}

Group C: Using economic intuition and the fact that $\Delta = 4ac - b^2$ and $\pi_{xx} = -2a$, discuss whether the SOC are satisfied.

Unconstrained Optimisation - ANSWERS

ACTIVITY ONE

Task One

Group A : Choose x and y to Maximise $\pi = px + qy - [ax^2 + bxy + cy^2 + K]$

Group B: $\pi_x = p - 2ax - by$ and $\pi_y = q - bx - 2cy$

Group C: $p - 2ax - by = 0$ and $q - bx - 2cy = 0$

Group D: $x^* = [2cp-bq]/[4ac - b^2]$ and $y^* = [2aq - bp]/[4ac - b^2]$

Then the sequence of answers is put together by the Lecturer and the whole group discusses the question " Are the values of x* and y* found by this procedure the firm's optimal output levels?" [Answer: We cannot be sure because the Second Order Conditions have not been checked] **.**

ACTIVITY TWO

Task One

Group A: -2a, -b, -2c, - b. Note that $\pi_{xy} = \pi_{yx}$, both being equal to – b.

Group B: $\Delta = 4ac - b^2$ and $\pi_{xx} = -2a$

Group C : It is immediately obvious that part of the SOC is satisfied because π_{xx} is clearly negative. But what about Δ? Economic intuition tells us that if the price of y, i.e. q goes up, it would be utterly perverse for the firm to sell less y. Similarly for x. In other words, we expect supply curves to be upward sloping!

From the solutions to FOC we note that $x^* = [2cp-bq]/[4ac - b^2]$ and $y^* = [2aq - bp]/[4ac - b^2]$. The impact of p on x* and of q on y* will be positive only if $2c/[4ac - b^2]$ and $2a/[4ac - b^2]$ is strictly positive. But 2c and 2a are clearly positive. Therefore in order for upward sloping supply curves, we require $[4ac - b^2]$ to be strictly positive. But $[4ac - b^2]$ is precisely the value of Δ. Hence, $\Delta > 0$ and the SOC are satisfied.

Constrained Optimisation - ACTIVITIES

ACTIVITY ONE

Learning Objectives

LO1: Students to learn how to obtain the FOC

LO2: Students learn how to solve the FOC

LO3: Students learn how to interpret the FOC

Task One

Students are given the following problem:

(a) Maximise $H= p_1X_1 + p_2X_2$ s.t . $G= X_1^2 + X_2^2 \leq r^2$ and $(X_1,X_2)\geq 0$

(b)Using your answer to (a) write down the FOC and interpret

(c) Using your answer to (c) solve these three equations

ACTIVITY TWO

Learning Objectives

LO1. Students consolidate solving and interpretation of FOC.

This is an individual activity. All students are given the following problem:

A small country is seeking to maximise its export revenue by selling its coffee and coconuts on the world market where they fetch \$ 6 and \$ 8 per ton respectively. In producing coffee and coconuts, the country is constrained by its production set which is given by:

$$X_1^2 + X_2^2 \leq 100$$

Where X_1 denotes coffee production in tons and X_2 denotes coconut production in tons.

Task One

Find the optimum production values X_1^* and X_2^* of coffee and coconuts and the export revenue earned. At what rate would the export revenue increase if the production set constraint was slightly relaxed.

Task Two

Re- solve the problem if the production constraint was changed to $X_1^2 + X_2^2 \leq 121$. By how much does export revenue change? How could you obtain this answer without resolving the whole problem? If the change in the production constraint to be shifted to $X_1^2 + X_2^2 \leq 121$ is achievable by buying a technology, what is the most the country should pay for this technology?.

ACTIVITY THREE

Learning Objectives

LO1. Students learn how to operate convexity conditions to check for a maximum

Task One

Students are asked to draw shapes of convex and non convex sets

Task Two

Students are asked to note common economic problems in which the relevant constraints do indeed form a convex set.

Task Three

Students are asked to note how to identify convex superior sets that arise commonly in economics.

Task Four

Students should check whether the convexity conditions are satisfied for the problem above.

Constrained Optimisation - ANSWERS

ACTIVITY ONE

Task One

(a) Answer: $L = p_1X_1 + p_2X_2 - \lambda [X_1^2 + X_2^2 - r^2]$

It is very important that students write the Lagrangean correctly, particularly noting the NEGATIVE sign between the objective and the constraint. They also need to note that in this problem r^2 is c.

(b) Answer:

$L_1 = p_1 - 2 \lambda X_1=0$ (1)

$L_2= p_2 - 2 \lambda X_2=0$ (2)

$\lambda [X_1^2 + X_2^2 - r^2]=0$

(c) Answer:

To solve these three equations we start with 1) and 2).

Rewrite (1) and (2) as:

$p_1 = 2 \lambda X_1$ (4)

$p_2 = 2 \lambda X_2$ (5)

Divide 4) by 5) to get:

$p_1/ p_2= X_1/ X_2$ (6)

From 4) and 5) since both p_1 and p_2 are strictly positive, it follows that (X_1^*,X_2^*) and λ^* are strictly positive.

Since $\lambda^*>0$, it follows that:

$X_1^2 + X_2^2 - r^2=0$. (7)

Now solve 6 and 7 to get:

$X_1^* = r\, p_1/\sqrt{[p_1^2 + p_2^2]}$ and $X_2^* = r\, p_2/\sqrt{[p_1^2 + p_2^2]}$

To find the value of λ*, substitute $X_1^* = r \sqrt{p_1^2/[p_1^2 + p_2^2]}$ in 4) to get:

$\lambda^* = \{\sqrt{[p_1^2 + p_2^2]}\}/2r$

Thus the full solution is:

$X_1^* = r\, p_1/\sqrt{[p_1^2 + p_2^2]}$ and $X_2^* = r\, p_2/\sqrt{[p_1^2 + p_2^2]}$ and $\lambda^* = \{\sqrt{[p_1^2 + p_2^2]}\}/2r$

As a final step substitute (X_1^*, X_2^*) from above into H to obtain:

$H^* = r\, p_1^2/\sqrt{[p_1^2 + p_2^2]} + r\, p_2^2/\sqrt{[p_1^2 + p_2^2]}$

ACTIVITY TWO

TASK ONE

This is just a numerical version of the problem analysed in Activity 1.

$L = 6X_1 + 8X_2 - \lambda\, [X_1^2 + X_2^2 - 100]$

FOC are:

(1): $L_1 = 6 - 2\lambda X_1 = 0$

(2): $L_2 = 8 - 2\lambda X_2 = 0$

(3): $\lambda[X_1^2 + X_2^2 - 100] = 0$

From (1) and (2), $¾ = X_1/X_2$ (A)

From (A) both X_1 and X_2 are strictly positive. Hence from (1) so is $\lambda > 0$ hence from (3)

$X_1^2 + X_2^2 - 100 = 0$ (B)

Substitute (A) into (B) to get:

($X_1^* = 6$, $X_2^* = 8$) and $R^* = 100$. Substitute $X_1^* = 6$ into (1) to get $\lambda^* = ½$.

The rate at which the export revenue increase if the production set constraint was slightly relaxed = $\lambda^* = ½$. $R^* = 100$

TASK TWO

If the constraint changes to $X_1^2 + X_2^2 \leq 121$, then one can re-solve the whole problem to get:

($X_1^{**} = 6.6$, $X_2^{**} = 8.8$) and $R^{**} = 110$. Hence $dR^* = 110-100= 10$.

Much easier to solve by using the information contained in the Lagrange Multiplier, i.e. $dR^*/d(r^2)= \lambda^*$

Since $\lambda^* = ½$, and $d(r^2)=121-100=21$, then $dR^* = ½ \times 21 = 10.5$ – which is approximately same as by long method.

ACTIVITY THREE

Task One

Convex sets are triangles, circles , semi circles, quarter circles, rectangles squares etc. The best example of a non-convex set is a doughnut.

Task Two

Consumer theory (budget set is a triangle, hence convex), Producer theory (production function with non-increasing returns to scale)

Task Three

Indifference curve shapes or iso-profit lines which are linear.

Task Four

Yes they are since $H= p_1X_1 + p_2X_2$ is linear and hence the "Superior" set S [H(,) ≥ h] for any h is a convex set. Furthermore since the Feasible set F is a quarter circle and hence convex.