  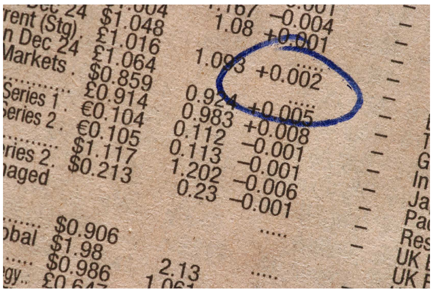

Teaching and Learning

Guide 6:

Non-Linear Equations

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Section 1: Introduction to the guide

The use of non-linear equations in teaching introductory mathematics in undergraduate economics degree courses, whether quadratics or higher order polynomials, tends to be limited and not given a great deal of time within the curriculum. Given the limited emphasis, it is perhaps unsurprising that students often fail to engage with non-linear equations with confidence or comprehend why there application in economics might be interesting. Indeed, it is perhaps fair to say that on occasions when students have mistakenly identified intercept points for a quadratic equation, perhaps the most common mistake students make in this topic, they appear unaware that they have provided answers that make no sense in terms of economics or indeed mathematics.

Similarly, we do not help ourselves as teaching professionals when we introduce the topic with bland and obvious statements such as “the world is not as simple as assumed in the linear model we have used thus far” and then plough straight into the maths. As teachers of the subject, therefore, it is incumbent on us not only to promote the use of non-linear equations and functions but to do so with a clear statement of why they matter within the study of economics. Given the relative balance of understanding of mathematical techniques versus economics, it is perhaps here, more than in any other aspect of mathematical economics, that clarifying by example is vital to improving understanding and engagement.

The following sections attempt to provide some guidance as to how this might be achieved. Key to this is the use of data and application to economic problems and use of economic principles to guide the mathematics. At the same time, many students do not have A-level maths qualifications, or if they do, they have not done maths for several years. However, many students do have a good grade at GCSE mathematics and so cohorts can present large variations from those that are not confident or are weak at maths to those that are able to confident with maths but simply did not choose to take an A-level in this topic.

Section 2: The Concept of Quadratic Functions

1. The concept of quadratic functions

The difficulties in moving the focus away from linearity are perhaps two fold. First, as teachers we have to undo the previous conception of a market or particular function and say that the real world is very unlikely to be linear and thus we must alter how to try to model and represent real world situations. Yet we need to do this in a way that does not lead the students to suddenly 'lose' all of the economic understanding they have built up from this simplifying assumption. This is not easy but is perhaps more straightforward than the second problem which is getting students to tackle non-linearity in a mathematical setting when they have only just grasped the rudiments of linear equations.

Clearly, the lecturer needs to build here on the assumption that linear equations have been taught and the students have learned how to use them, interpret them in an economically sensible manner and that they are comfortable when faced with problems that require solutions to linear equation models. If you are uncertain about the students grasp of linearity, then please refer to Teaching Guides 2 and 3 on this subject before reading any further.

In many respects, the difficulty with non-linearity is that students rarely encounter it in the early years of their undergraduate degrees (if at all) and thus meaningful application is not always forthcoming in parallel subject teaching. Again this reflects the more difficult application of non-linear models within undergraduate teaching although that does not mean it is never done and nor does it mean it has no value. This does not mean they do not realise that the assumption of linearity is likely to be invalid, but that we do not do enough to show why it is useful, how conclusions change and perhaps as importantly why assuming linearity is not always helpful.

2. Presenting the concept of quadratic functions

One approach is use a simple excel worksheet to demonstrate visually what happens when each of the parameters of a quadratic equation are changed. Lecturers could plot the values of x and y for a given quadratic equation ($y = a + bx + cx^2$), perhaps one chosen by the students in a lecture or small group and then illustrate this function. Values of a, b and c in the equation can be altered and the equation re-plotted against the original graph.

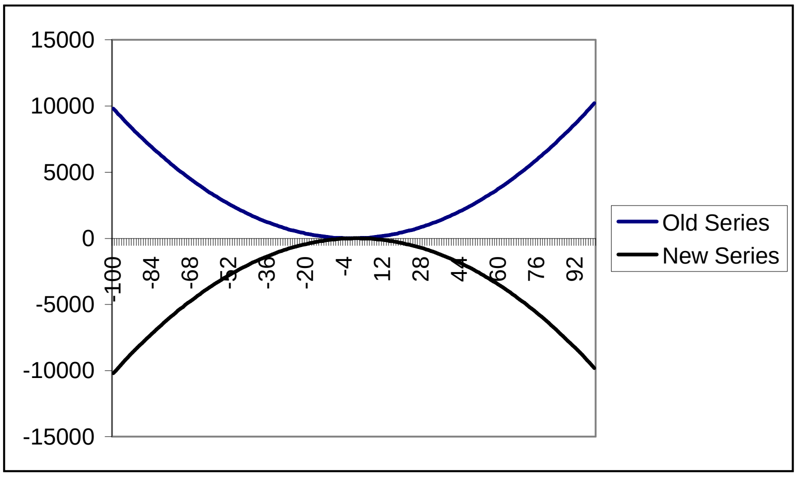
It is helpful to begin by switching between positive and negative versions of the same number before altering the absolute value of the number. This demonstrates very quickly to students that changing the value of *a* shifts the line up and down, it changes the value at which the line cuts the y axis (x=0). Changing the value of *b* shifts the line left and right.

Finally and most importantly switching the value of *c* from positive or negative gives the line its ‘u’ or ‘n’ shape, while increasing or decreasing the absolute value of *c* makes the slops steeper or shallower. In the sample below we show the effect of a change in the c parameter from 1 to -1. These worksheets could be mounted on the module website to allow students to work with in their own time.

Function: $y = a + bX + cX^2$

Values	a	b	c
old values	3	2	1
new values	3	2	-1

X	Old Series	New Series
-100	9803	-10197
-99	9606	-9996
-98	9411	-9797
-97	9218	-9600
-96	9027	-9405
-95	8838	-9212
-94	8651	-9021
-93	8466	-8832
-92	8283	-8645
-91	8102	-8460
-90	7923	-8277
-89	7746	-8096
-88	7571	-7917
-87	7398	-7740
-86	7227	-7565
-85	7058	-7392
-84	6891	-7221
-83	6726	-7052
-82	6563	-6885



3. Delivering the concept of quadratic functions to small or larger groups

Students could benefit from a kinaesthetic and visual method of delivery. Lecturers could create a range of laminated cards which cover the values 0 -1- inclusive, the variables x and y and the power ‘2’. Students select cards and create their own quadratic in front of them and in pairs they try to sketch what their quadratic might look like. Students could look at the quadratics and sketches from other pairs.

The lecturer could select some or all of the quadratics and using Excel project what the functions actually look like. This could be simple, effective and ‘instant’ way for students to develop a feeling for what a quadratic looks like and the function is constructed and behaves according to its constituent coefficients.

Links with the online question bank

Questions on quadratic functions can be found on the METAL website at:
http://www.metalproject.co.uk/METAL/Resources/Question_bank/Algebra/index.html
Lecturers might find it useful to refer to the questions on adding polynomials at
http://www.metalproject.co.uk/METAL/Resources/Question_bank/Algebra/index.html as a precursor to the subsequent material or cubic and polynomial functions. These questions could also be used to further differentiate teaching and learning.

Video clips

Although there are no clips which deal specifically with quadratics, lecturers might want to look at the clip 1.09 at http://www.metalproject.co.uk/Resources/Films/Mathematical_review/index.html which examines powers and indices in the context of gold mining and oil exploration.

4. Discussion Questions

Ask students to think of any relationships which are u-shaped. Could students who follow geography courses make links e.g. mathematically modelling u-shaped valleys or river beds? Obvious links to the natural world abound e.g. the wear on a step or the trajectory of a thrown tennis ball. Students could see how many quadratic relationships they could identify and describe within a week. Higher ability students might want to reflect on **why** the relationship is quadratic as opposed to say a linear function.

5. Activities

Learning Objectives

LO1: Students learn to calculate values of y for different values of x using quadratic functions

LO2: Students learn how to tabulate and plot quadratic functions

LO3: Students learn that many economic relationships are rarely linear and some can be expressed using a quadratic expression.

Task One

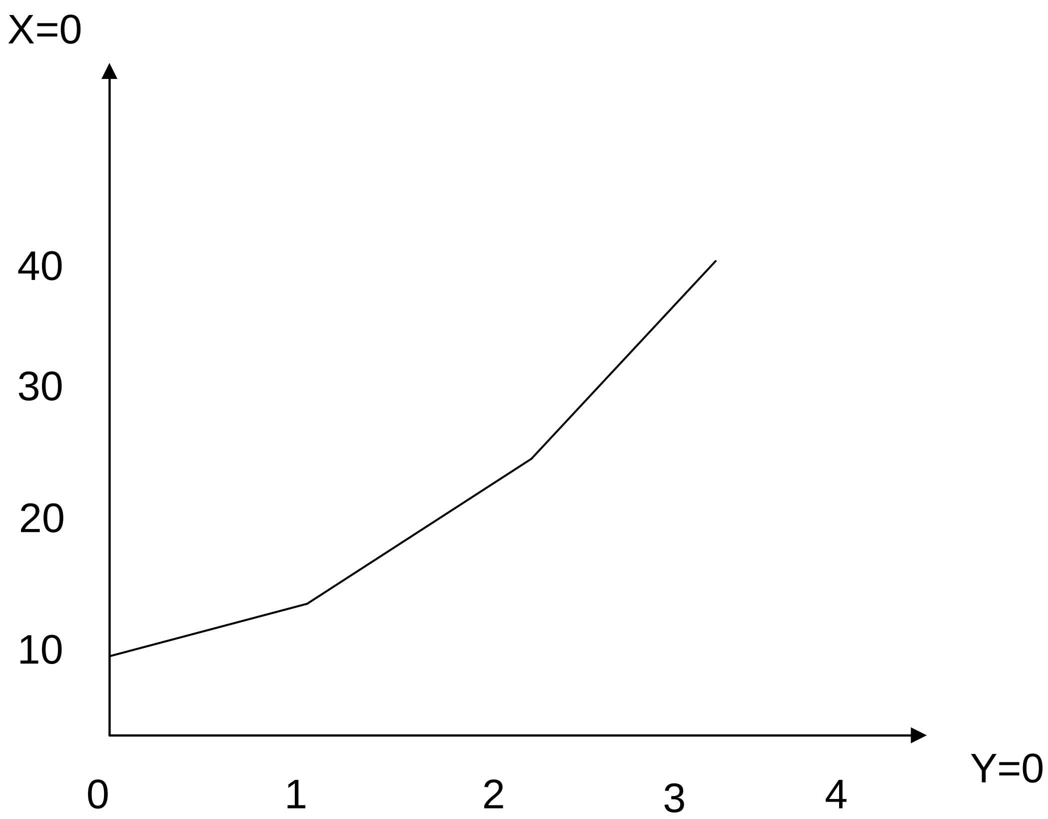
To begin with, let us assume that we are in a small group setting, such as a tutorial or problems class, and we will show later how the activity can be adjusted for the large group (lecture theatre) setting.

The first stage is to split the group into pairs or threes, which can be done with the people sitting next to each other – there is no need to enforce randomness on this procedure as in fact that might be threatening for some students and again create barriers to learning. The second stage is to provide each group with a piece of graph paper with a relatively simple non-linear function written onto it (e.g. $y = 10 + 4x^2$).

Each group could have the same function or you could vary them so that there is exposure to a number of equations. Another possibility is for the students to be given a series of equations and they chose one of them to examine, which puts the element of control into their hands, again helping to create a more positive atmosphere around the learning.

The exercise they are then set is to calculate the change in the dependent variable (y) for the same (or possibly different) changes in the independent variable (x). These can be written down in tabular form on the sheet and then they can be plotted. Thus for example, for $y = 10 + 4x^2$ it might look like this:

(x)	(y)
0	10
1	14
2	26
3	46



The outcome of this exercise is that it should be apparent that for a given move from one value in (x) the change in (y) varies and thus the previously simple relationship between (x) and (y) found in linear equations no longer holds. Crucially too the impact of changes in (x) on (y) depends on where you start with (x) – a low value or a high value for instance.

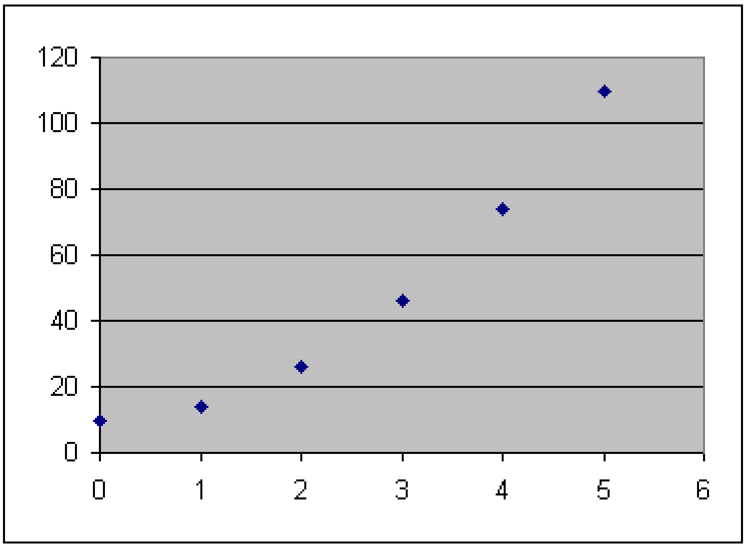
Another issue here of course is how the lines are plotted. At first it is best to make explicit that you simply want to plot the points as straight lines between two pairs of co-ordinates. Once this has been done then, discussion of what happens to all the values in between the whole numbers we have chosen and that can lead to plotting a smooth curve. This reinforces the staging process from linear models to non-linear as the shape becomes exacerbated.

In essence, the exercise builds on prior learning as it requires manipulation of two variables to find solutions to an equation and then plotting these on a graph, both of which they will have done previously. In a large group setting the sheets can be handed out at the start of the session and students can work on their own. Values for x are provided and they have to find y. They then offer their answers to the lecturer who has a plot on the whiteboard or computer at the front of the lecture theatre. As answers are plotted the non-linear nature of the relationship should become apparent quite readily. This can be done using Excel and plotting from the data therein which could be pre-loaded with more data points than they provide in responses but which allow for the curvature of equations to be emphasised.

An Example of an Excel chart with Data:

y= 10 + 4x2

x	y
0	10
1	14
2	26
3	46
4	74
5	110



Task Two

Given the demand function $Q_D = 75 - (1/4)P$

a. Find the total revenue function written in terms of Q [TR = *P*(Q)]
b. Calculate the point at which total revenue is equal to zero
c. Calculate the point at which total revenue is maximised.

Task Three

A manufacturer faces two types of costs in its production process, fixed costs which are equal to £1000 and variable costs which are equal to £2 for each tyre produced.

a. State the total cost function for this firm
b. Calculate average costs
c. Calculate total costs if 700 tyres are produced.

Task Four

Using the total revenue function calculated from question 1 and the total cost function calculated in question 2.

a. Derive the profit function for the firm
b. What is the value of profits when production equals 125 units?

Task Five

A company discovers the following economic information about its costs and demand function:

Demand Data	***Cost Data***
$Q = 33 - \frac{1}{2}p$	Fixed costs are £200
	Variable costs are £8 per unit

i) Derive the profits function for the firm
ii) What is the breakeven quantity?
iii) At what output level would profits be maximised?

ANSWERS

Task One

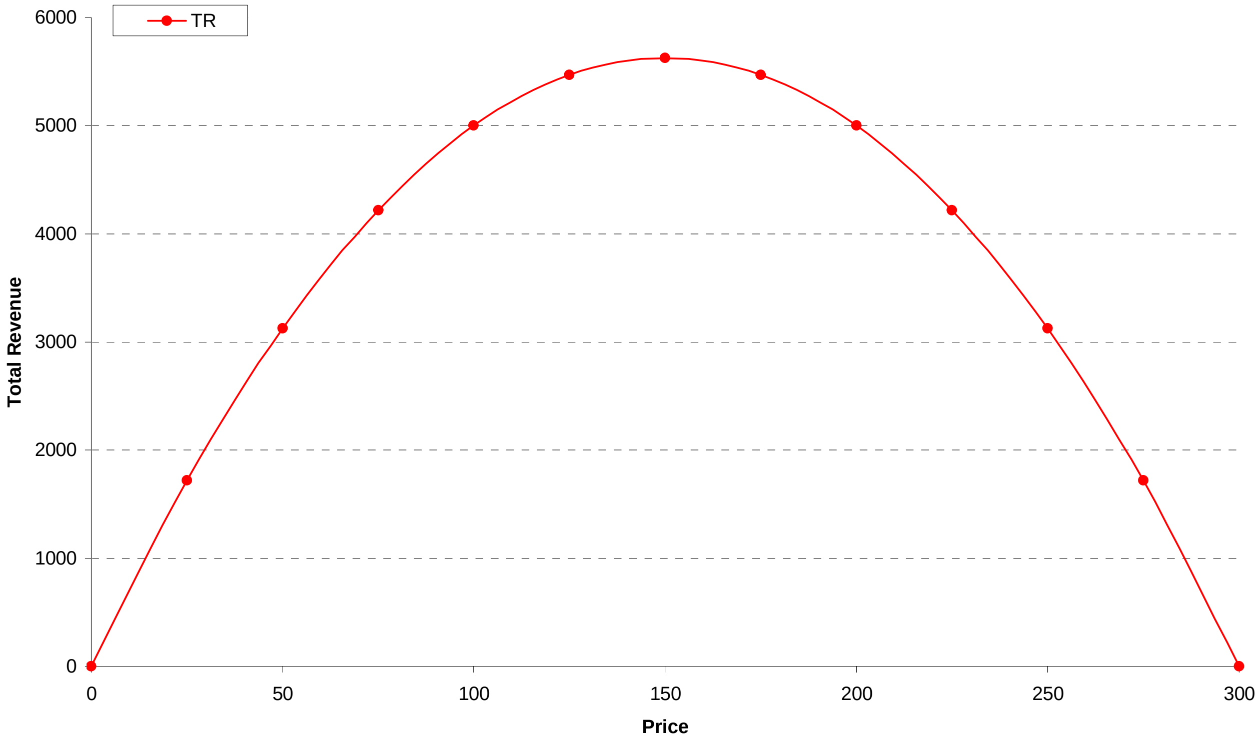
See Task

Task Two

a. $TR = 75p - \frac{1}{4}P^2$

b. TR=0, p=300

c. TR is maximised when P=150 (see diagram below)



Task Three

Let *t* denote the number of tyres produced

a. TC = 1000 + 2*t*

b. AC = TC/*t* = $\frac{1000}{t}+2$

c. TC=£2400 when *t*=700

Task Four

a. Profit = TR-TC = $300Q-\frac{1}{4}Q^2-1000-2Q$

b. Profit = -£26,250 i.e. a loss

Task Five

i)

$$Q = 33 - \frac{1}{2}p$$
$$\Rightarrow p = 66 - 2Q$$
$$TR = p \times Q = 66Q - 2Q^2$$
$$\pi = TR - TC = 66Q - 2Q^2 - 200 - 8Q$$
$$\pi = -2Q^2 + 58Q - 200$$

ii) Breakeven occurs when π=0

$$\pi = -2Q^2 + 58Q - 200 = 0$$

Factorising we get:

$$(-2Q + 50)(Q - 4) = 0$$

Q=4 or Q=25

So, the firm breaks even when output is 4 units and also when output is 25 units

iii) Profits maximised $\frac{\partial \pi}{\partial Q} = 0$

$$\pi = -2Q^2 + 58Q - 200$$
$$\frac{\partial \pi}{\partial Q} = -4Q + 58$$

$$\frac{\partial \pi}{\partial Q} = 0$$
$$-4Q + 58 = 0$$
$$Q = \frac{58}{4} = 14.5 units$$

6. Top Tips

One of the best ways of keeping students engaged with non-linear ideas is to use as often as possible graphical representations to illustrate points especially when dealing with more complex equations. A visual cue often helps a more intuitive understanding of the ideas.

It often helps to approach quadratic equations in a structured way such as the following:
1. Determine the basic shape using the coefficient on the squared term.
2. Find the y intercept by substituting in x=0.
3. Find the x intercepts (if possible) using the formula.

7. Conclusion

Create opportunities for students to use both graphical and algebraic representations to help them really acquire a solid sense of what a quadratic function looks like and how it is composed. This would provide a good foundation for the next section.

Section 3: Cubic and other polynomial functions

1. The concept of cubic and other polynomial functions

Higher order polynomial functions, such as cubic, share some of the problems of quadratic equations in that they are difficult to visualise and the equations look “messy”. They do not however, typically create problems and issues to the student that are distinct. It is also generally the case that we actually offer very little in the way of mathematical tools to deal with them. We rarely go beyond simply plotting them for different values of x. The mathematical tools applied to them, such as differentiation, are covered in a later section of this guide.

Compared to linear and quadratic functions higher order functions have few properties that are particularly useful in describing economic concepts, perhaps explaining why we often spend so little time with them. This guide does not therefore spend a long time explaining why a cubic cost function is of particular economic relevance for example.

2. Presenting the concept of cubic and other polynomial functions

Students will be interested to see real world examples of cubic and polynomial functions from the world of economics and business. Empirical studies such as the business and Kondratiev cycles can engage students. Starting with a 'real world' introduction can help to switch students on instead of launching straight into the mathematics with the practical context tagged on at the end.

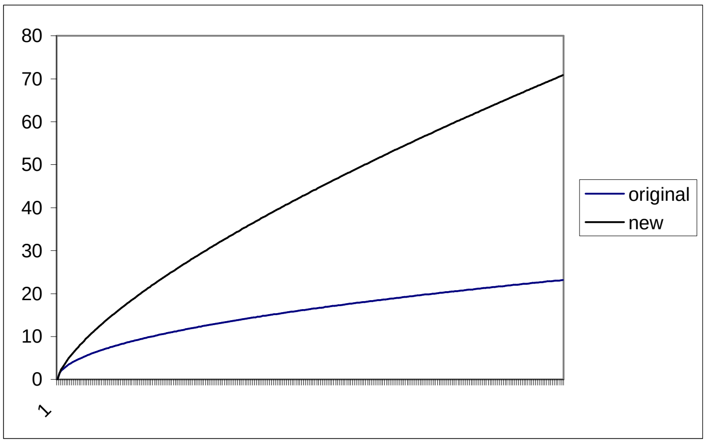
3. Delivering the concept of cubic and other polynomial functions to small or larger groups

The Cobb-Douglas production function can be a good way to look at this higher level material initially by concentrating on the short-run, i.e. with one input fixed. The topic is introduced generally, usually with reference to the location of the same material in the microeconomics course.

Lecturers could then discuss the short versus the long run and then give a specific example for the Cobb-Douglas case. Finally, students could see what happens to output when we alter the size of the elasticity parameter on the variable factor. We use this to introduce the economic concept of diminishing marginal returns mathematically (before returning to it with differentiation later in the course).

Again providing a visual point of access to this topic is of use. Below is an example of a simple excel worksheet that describes the relationship between output and capital input. The slope of the line changes as the value of diminishing marginal returns to the capital input changes (in this case from 0.5 to 0.7).

Diminishing Marginal Returns					
	0.5	0.5	0.7		
L		Old Series	New Series	original	new
	2	0	0	0	0
	2	1	1	1.414214	1.414214
	2	2	2	2	2.297397
	2	3	3	2.44949	3.051405
	2	4	4	2.828427	3.732132
	2	5	5	3.162278	4.363088
	2	6	6	3.464102	4.957022
	2	7	7	3.741657	5.521838
	2	8	8	4	6.062866
	2	9	9	4.242641	6.583923
	2	10	10	4.472136	7.087858
	2	11	11	4.690416	7.576871
	2	12	12	4.898979	8.052706
	2	13	13	5.09902	8.516778
	2	14	14	5.291503	8.970252
	2	15	15	5.477226	9.414102
	2	16	16	5.656854	9.849155
	2	17	17	5.830952	10.27612
	2	18	18	6	10.69561
	2	19	19	6.164414	11.10817
	2	20	20	6.324555	11.51426



There are several discussion questions below which are designed to place quadratics and polynomials in an applied economics context.

Links with the online question bank

As noted above, questions on quadratic functions and on adding polynomials can be found on the METAL website at:

http://www.metalproject.co.uk/METAL/Resources/Question_bank/Algebra/index.html

Video clips

Although there are no clips which deal specifically with polynomials, lecturers might want to look at the clip 1.09 at http://www.metalproject.co.uk/Resources/Films/Mathematical_review/index.html which examines powers and indices in the context of gold mining and oil exploration.

4. Discussion Questions

The learning outcomes from these discussion exercises could be seen as follows:

LO1: Raising awareness of non-linearity through empirical analysis

LO2: Direct student engagement with the subject by prior assignment of tasks

LO3: A more smooth transition form linear to non-linear forms that assists understanding

(a)The Laffer Curve

Students could consider the Laffer curve and in the group you can raise the question of why it is that when tax rates rise government tax revenues do not always rise. Students might discuss work effort or the growing black economy and it is here that you can then start to tease out the non-linearity that exists in the world. Indeed, you could consider asking the students to gather data and evidence of tax rates and revenue returns from a given OECD economy to see how the two relate. In this way you are not only making explicit the link between the quantitative element in a course to the economics, but you are also giving the students a sense of “ownership” of the topic as it is they who are gathering the evidence and it is they who will work to explore the relationship that will lead into a discussion of non-linearity. It is an important feature of getting engagement by making the students active prior to the session rather that they simple turn up and be told what non-linearity is in a receptive or passive mode.

Introduction: A key concept in economics is of course the way agents make decisions and what impact those decisions have not only themselves but on other agents too. In this case, the focus is on the important relationships between government tax policies and tax revenues via the work effort in the economy. In discussing this area it should become apparent to the students that in making what are apparently straightforward choices as a government can in fact be a process fraught with danger as the relationships between tax rates and total revenues are non-linear.

Discussion: There could be quite a neat if somewhat controversial way of opening this debate and that is to suggest that the students consider that they are the Chancellor of the Exchequer and that due to government pledges in the election, they have to find more money to pay for higher education. Again, as mentioned above, this could require two or three students (depending on class size) to gather some data for a given country and bring them along to the class. They can then show this is an appropriate form to the rest of the group (such as tabular or graphical representation for example) before then highlighting what they have found. Here they have to, in effect, describe patterns in data and show, where relevant, when policy changes occur, or at least relate policy to outcomes.

Before such a presentation takes place though, the class as whole could be posed a series of questions about what they might expect to see. This could begin with a discussion of personal tax rates in the particular country chosen and how these compare with what they believe tax rates are in the UK as a means of engaging them directly in the topic but also to consider background information. (See for example the Financial Times of 18th July 2006, “Bush Believes Higher Revenue Vindicate Tax Cuts”). Then they could be challenged to say what they expect to happen if the top rate was increased by 10%, 20% or other values you decide to work with. This sets some hypotheses that can then be “tested” using some data from the real world.

Form this initial discussion, which need only take 10-15 minutes or so, the key finding of non-linearity can be exposed and then this can lead to the mathematical representation of such a relationship. Here the work can build directly on the linear approach and can show how a simple equation that they are used to such as $y = \alpha + \beta x$ will not provide an accurate means of explaining the relationships they have been discussing. Showing a number of different ways to do this can lead into quadratic and higher order polynomial equations which they can then be expected to work on as examples.

Conclusions: In total then, the students will be getting an insight into data-led investigation, some exposure to the skills and approaches in the presentation of an argument and a clear link between the economics and maths of non-linear equations. As follow up work, beyond the mathematical exercises they will undertake in using and manipulating non-linear equations, you could get the students to consider the original work of Laffer and then perhaps one or two of the empirical pieces of research work on the US economy in the 1990s as to whether the Laffer view holds or not.

(b) Economic Convergence

A key policy debate is whether developing economies can “catch up” with income levels in more developed economies, and in some cases whether they will ever surpass those in the richest countries. While many alternative examples could be used to illustrate this point it useful to centre the debate on a country such as China (and India) which has experienced recent rapid growth and has featured heavily in the press as a result. The topic therefore is also useful for testing their knowledge of current affairs.

The topic is relevant because the income paths of many countries display the properties of a quadratic function when expressed relative to income levels in the income frontier (usually taken as the US). The topic can be used to reinforce a number of points and mathematical skills. For example: assuming linearity would be misleading and would provide policy prescriptions that would be incorrect and/or costly. It also uses mathematical skills such as calculating intercept points with the x-axis and calculating points on a quadratic function.

These examples rely heavily on real world data which for this topic have several attractive features. These include:

- The income path of the US has been growing at a constant rate for 170+ years (Jones, 1995, Quarterly Journal of Economics). It therefore has the properties of a straight line over this period.

- The European countries and Japan, whose incomes caught up over the post war period, have the features of a quadratic function, indeed there has even been some relative decline over the 1990s.
- There are several comparator countries within the same region that might be used to provide possible income paths into the future (South Korea, Hong Kong, Malaysia etc.). These might be described as on the first part of a quadratic function

This discussion topic works well in a tutorial/seminar group. The data might be collected from a number of sources including the World Bank and Penn world Tables.

Introduction: This might be done either using an article from the Financial Times (Martin Wolf has written on this topic in the past) or from the Economist or by asking if anyone knows the growth rates for China. From this a discussion might be held as to whether they think income levels will catch-up with those in the US or surpass them and how long it might take.

Discussion: From this you could then turn to the data (which you could either graph for them and present as handouts, as slides in a presentation or actually given them in spreadsheets and look at them "live"). As the income path of the US can be represented by a straight line you might find the straight line that fits the Chinese data (the slope of which will vary depending on the time period you choose). You could then pose the question of when will China overtake the US if it keeps growing at this rate. This is just the intersection of 2-lines. This also shows that any "leap-frogging" that might take place is a long way into the future.

The discussion could be brought forward by asking whether an assumption of linearity is plausible. The post 1945 experience of European countries, Japan or the South East Asian tiger economies might be used as examples here, in particular when graphed relative to the US. The quadratic functions for these countries might be provided to them. The quadratic income paths of these more developed countries could then be applied to understand catch-up and leap-frogging in China, assuming they were to follow the same development path. If the relative income level equation is reversed (country X / US) then students could be asked to calculate whether the quadratic passes through zero (indicating when the income gap to the US is zero) if at all (incomes converge to a point below US incomes). If it does pass through zero, if students are given current income levels they could be asked how many years this will take (the value of x when y equals zero). These tasks could be done as individuals or each student/pair could take a different country.

Conclusions: The topic could finish with a discussion of whether these scenarios are plausible and what factors it might rely on. Or if each student takes an individual country they could be asked to comment whether they think the path described by their country might apply to China. The links to growth theory specifically and macroeconomics more generally could be made clear.

(c) The Liquidity Trap

Introduction: Finally, in a similar fashion to Topic 1, we offer another macroeconomic example to show how policy changes do not always generate the intended outcomes. There are two possible examples that can be used to illustrate the liquidity trap – the case of the UK in the 1930s and here a discussion could centre on the Depression and the rise of Keynesianism or more recently the deflation experienced in the Japanese economy and the way in which government attempts to stimulate demand failed for over a decade.

Discussion: Whether the case chosen is the UK or Japan (or indeed any other economy!), the exercise can be set up in a similar fashion to the Laffer curve exercise above. Two or three of the class can be asked to gather some data relating to the economy in 1930s Britain or modern Japan and then to present it to the rest of the class as a briefing in economic history. They need to focus though on the level of investment and the rates of interest charged in both cases. Another way to illustrate the same cases is with some newspaper articles that deal with these points in time or reflect them historically. This can generate discussion of policy and how impotent the policy instruments can be depending on the context in which you change them. For example, a contrast could be made with the relatively low levels of interest rates in the UK and the associated investment levels that can be shown to be rising, with the very low rates of investment in the 1930s despite record low interest rates. Getting students to see how the relationship is not the same is a starting point for consideration of non-linearities in economic relationships.

Prior to the presentation though, you could rehearse the arguments about how agents respond to changes in interest rates and what we might expect to happen to investment levels. A nice linking point here is to the IS-LM model and the validity or otherwise of the investment relationship it assumes.

Once the presentation has been made, the key point of a lack of response in investment to changes in very low levels of rates of interest can be highlighted. The point can be made that if we think about economic theory, we often imagine a simple tweak of policy variables having a defined and constant effect on other variables in the economy. The two examples provided would show that not to be the case.

The steps then would be to take this unexpected non-response policy and start to expand on it by exploring how investment does in fact respond to changes in rates of interest but in a non-constant (i.e. non-linear) fashion. The nature of this relationship could then be explained in terms of quadratic or higher order polynomials depending on whether real examples are used to illustrate the point or not. This in turn leads into the more general issues surrounding non-linear equations and how to solve them.

Conclusion: The key outcome here is that the simple (naïve?) link between a policy and its potential outcome effects is broken and as such challenges the student to explain why this might happen. In doing so, the student needs to be able to explain non-linearity in a relatively intuitive manner which is often a good indicator of understanding of the material.

5. Activities

Task One: Battleships

Despite the obvious benefit of sheets of practice questions for promoting understanding and exam preparation by students it can often be the case that this preparation is not done, in particular if problem classes simply work through the answers. Providing an incentive to students to complete the tutorial sheets is therefore of use.

In battleships, students are split into teams and given a reasonably long list of questions that they must work through, as well as their battleship grid and battleships at the end of the previous tutorial. In time for the next tutorial it is the task for each team to work through the questions on the list in their teams and decide where they will locate their battleships. The game is played as normal battleships but to get a go at hitting the opponents ships they must have answered a question correctly. It can help to call question numbers from the list randomly, even if questions are structured to form part of *bigger* questions, so there is a mix between hard and easy questions. Students can get very involved with such a game.

Task Two: Leontief Production Functions

Students do sometimes fail to recognise that production functions are not sacrosanct and that there are many versions of production function that firms can apparently work from. Indeed, the very fact that different forms exist of the same type of function is often a revelation to some students! So the notion that there may in fact be non-linearities in production functions is probably a very big step for some students to take in terms of their comprehension of the links between maths and economics.

Students are asked to consider the Leontief case which has the virtue of being linear around a break-point that makes the whole function non-linear. It is an extreme case of production function but does illustrate the issue of non-linearity quite nicely. To do this, you could get the students to consider themselves as the local council which is charged with keeping streets clean. They are a low-tech council and rely on two factors of production; workers and brooms. The discussion can then centre on how best to combine workers and brooms to sweep the streets of whichever district you care to choose – maybe a very messy student one!

Using either graph paper or indeed spreadsheets in Excel the students can draw up simple table of output (i.e. clean streets) in relation to inputs on the basis that one worker with one broom creates two clean streets for example. In this way you can then show how everything is constrained by the factor that is least abundant.

Thus we have 20 brooms but only 6 workers we only achieve output of 12 clean streets which is the output associated with 6 workers and so forth. As a change, you could then suggest that the council now invests in new technology by buying a machine to replace the brooms and the output relationship now is one worker plus one machine equals 6 clean streets. Before repeating the exercise, see what the students expect to happen to the isoquants and then explain the outcome they find.

While this does not give us the curved non-linearity of many of the other exercises in this area it does highlight how production functions generate some unusually shaped isoquants and the difficulties this provides for finding the optimal combination of factors of production.

Task Four: Production functions, Growth and Economic Development

This topic works well as an extension of the discussion topic on economic convergence described above. The growth of output is explained by the growth of inputs into the production function. The relative importance of these inputs in a Cobb-Douglas production depends upon the value of the elasticity parameters on each of the inputs. As a result the policy can change. This debate is well known in the growth accounting literature and has been discussed in particular with respect to the growth in the South East Asian economies. As in that literature, this debate works best with respect to the share of physical capital investment versus technological progress/productivity in growth because productivity is measured as the residual.

This exercise offers another way of introducing the idea of production functions, output is made up of the inputs into the production function. Assume a production function of the form, $Y = AK^{\alpha}H^{\beta}$, where Y is output, K is physical capital, H is human capital and A is productivity/technology. The basic idea of the exercise is to demonstrate that for a given rate of increase in the inputs their relative importance in describing output varies. A discussion should also be made here about why productivity cannot be measured and is therefore calculated as the residual.

This is most easily done if the production function is in log form so that $\ln A = \ln Y - \alpha \ln K - \beta \ln H$. Get them initially to work out an initial value of output for a given set of inputs and parameter values, it is easier if they are given the log values of the inputs. They could then work out a final value and be asked what percentage of the change in output was explained by human capital and physical capital and what by productivity.

China	Output per capita	Physical capital α=0.6	Human capital β = 0.4	Productivity
1990	6.4	8.3	2.6	0.38
2006	7.36	9.59	2.91	0.44

Note: All values are logged.

Then you could change the value of the parameters and repeat the exercise, say decreasing the alpha parameter to 0.4. This sometimes works best if one of these iterations is done collectively and they are them broken into small groups and asked to repeat the exercise for the new parameter set (different parameters could be given to different groups). A discussion could then be held as to which input, human capital, physical capital or productivity governments should focus on.

Reference could be made here to the recent UK Budget which discuss these issues. This could be finished off with a discussion how diminishing marginal returns would mean we would not want to accumulate one input alone. While it is helpful to use the logged version of the function it is important that the students are reminded of what the normal Cobb-Douglas production function looks like during this task.

6. Top Tips

This material could be made more accessible by students looking back over earlier material on logarithms and powers. The online question bank could be used for indices at http://www.metalproject.co.uk/METAL/Resources/Question_bank/Algebra/index.html

7. Conclusion

This material should be 'built up' in stages starting with an introduction to non-linear functions and their prevalence in business and economics. Quadratics can probably be covered fairly quickly but care should be taken to ensure students have a sound grounding before they embark on cubic functions and polynomials. A recurring theme in this guide has been the need for lecturers to ensure that all of the theory and concepts are contextualised and, wherever possible, with students exploring 'real world' or even personal examples to help consolidate their learning.