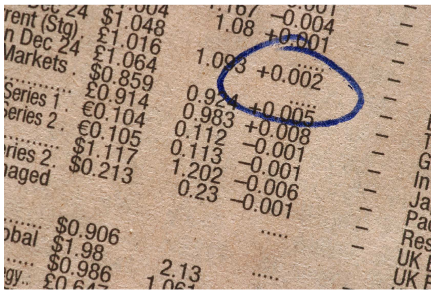
  

Teaching and Learning

Guide 5:

Finance and Growth

Table of Contents

Section 1: Introduction to the guide

This Guide is designed to set out some of the basic mathematical concepts needed to teach financial economics at undergraduate level. The concepts covered by this guide are:

- Arithmetic and geometric series;
- Simple and compound interest;
- The time value of money;
- Investment appraisal; and
- The NPV rule and the IRR rate of return rule.

The modern day finance lecturer is fortunate in the number of real world examples they have available to them. This Guide provides a number of current examples covering borrowing, investment and annuities to support the delivery of core concepts. The lecturer is encouraged to revisit the sources cited to update the examples nearer the time of delivery.

The use of Excel is an essential tool for anyone working in finance. Throughout this guide Excel screenshots and links to files are provided. It would be useful therefore if the session utilising this material were presented in a classroom where students can gain hands on experience.

A large part of this guide is devoted to “Teaching and Learning” exercises. Years of teaching this discipline have led this author to believe that this is definitely a “doing” subject and with lots of hands on practise the concepts should be easier to understand. This guide utilises a number of mathematic concepts explored in other guides in the series, namely: (i) the use of the summation notation; (ii) use of natural logarithms; (iii) use of the exponential function; (iv) the quadratic equation; (v) finding the equation of a straight line.

Section 2: Arithmetic & Geometric Sequences and Series

1. The concept of Arithmetic & Geometric Sequences and Series

Many students will have little idea exactly what is meant by a 'mathematical series' and it is likely that fewer still will have a grasp that mathematics can be used to help solve problems regarding finance and growth. Lecturers might find it advantageous to start this material with a simple and clear explanation of the terms, "finance", "income", "wealth" and "growth". For example,

'Finance' refers to a wide range of economic activities which are concerned with the management of income and wealth.

'Income' can be defined as income is defined as a stream of payments which are received in return for providing something: Apple Corporation for example received huge incomes from selling more than 14 million of its iPod personal music players in the last three months of 2005.

'Wealth' on the other hand is a 'stock concept' and refers to the accumulation of valuables or assets over time e.g. the Russian businessman Roman Abramovich has an estimated wealth of around £10 billion[1].

'Growth' means the increase (if it's positive) or decrease (if it is negative) in the size of something over a given period of time e.g. during the Summer of 2007 it was reported that Crops in the Black Sea area of Europe, were ruined by bad weather with Chinese production of wheat expected to fall by 10% as a result of both flooding and droughts. These examples will help students to appreciate that series are about **systematic** changes in economic variables over time. Students could be encouraged to research their own stories regarding growth.

2. Presenting the concept of Arithmetic & Geometric Sequences and Series

Students will need to be clear about the difference between an arithmetic progression and a geometric progression. The labels themselves are opaque and probably quite off-putting so a restatement is crucial.

The following could be useful introductory presentation:

Arithmetic Progression

An arithmetic series (or progression) is a series in which there is a ***constant difference*** between each term. For example:

100, 101, 102, 103, 104….

3, 7, 11, 15, 19….

The distinct feature of these series (or sequences) is that each term, after the first, is obtained by adding a constant, d, to the previous term. In the examples above, 'd' is 1 and 4 respectively. In the discussion that follows 'a' is used to represent the first term.

Each element of a series (or sequence) can be identified by reference to its position in the sequence (or its term number). For example, in the second series above, the first term, T_1, of the sequence is a = 3 and the second term, T_2, of the sequence is a + d = 3 + 4 = 7. Hence the value of any term can be determined with reference to the values for 'a' and 'd' and its position in the sequence.

This is best illustrated in tabular form:

Value	**3**	**7**	**11**	**15**	**..**	**..**
Sequence	a	a + d	a + 2d	a + 3d		a + (n-1)d
Term number	T_1	T_2	T_3	T_4		T_n

Equally, the overview of a geometric progression would also need to be provided:

Geometric Progression

A geometric series (or progression) is one in which there is a ***constant ratio*** between any two geometric terms. For example:

1, 10, 100, 1000, 10000,….

[1] Source: The Times Rich List 2006

3, 9, 27, 81, 243,….

The distinct feature of these series is that each term, after the first, is obtained by multiplying the previous term by a constant, r. In the examples above, ‘r’ is 10 and 3 respectively. Again, in the discussion that follows ‘a’ is used to represent the first term.

Each element of a sequence can be identified by reference to its position in the sequence (or its term number). For example, in the second series above, the first term, T_1, of the sequence is a = 3 and the second term, T_2, of the sequence is ar = 3 x 3. The third term, T_3, is $ar^2 = 3 \times 3^2$. Hence the value of any term can be determined with reference to the values for ‘a’ and ‘r’ and its position in the sequence.

In tabular form:

Value	**3**	**9**	**27**	**81**	**..**	**..**
Sequence	a	ar	ar^2	ar^3		$ar^{(n-1)}$
Term number	T_1	T_2	T_3	T_4		T_n

3. Delivering the concept of Arithmetic & Geometric Sequences and Series to small and larger groups

Students could benefit from watching one or more clips regarding geometric series (See Video clips immediately below) and then watch a clip on their own or in pairs and then report back to the group:

- What the clip was about;
- What the **economic** issue was ie. What was the presenter trying to explain?
- How geometric series could be used to solve the problem or issue.

Links to the online question bank

This Guide draws upon complementary knowledge eg. logarithms, fractions etc and so students might wish to practise these underpinning concepts at
http://www.metalproject.co.uk/METAL/Resources/Question_bank/Algebra/index.html
and http://www.metalproject.co.uk/METAL/Resources/Question_bank/Numbers/index.html

Video clips

A useful general clip on the practical application of series, growth and investments can be found at: http://www.metalproject.co.uk/Resources/Films/Mathematics_of_finance/index.html under 3.08 “Eurotunnel - A Bad Investment”. This could to ‘set the scene’ for this topic.

There are several useful clips dedicated to geometric series. These can also be found at http://www.metalproject.co.uk/Resources/Films/Mathematics_of_finance/index.html.

4. Discussion

Students could be asked to consider ways in which other variables or phenomena can grow or change over time. For example, how bacteria grow or how a forest fire can appear to devour parkland with an increasing speed. This could be linked to economic issues such as how stock markets can respond with ‘increasing pessimism’ as result of expectations – eg. the Wall Street Crash – and the notion of the rate of change becoming more rapid.

5. Activities

Learning Objectives

LO1: Students to understand the meaning of ‘geometric series’

LO2: Students to learn how to calculate a geometric series and the constant ratio

LO3: Students to learn the meaning of ‘compound interest’

LO4: Students to learn how to independently calculate compound interest.

ACTIVITY ONE

Background and Worked Example

One of the applications of geometric series is the calculation of compound interest. Here the sum on which interest is paid includes the interest that has been earned in previous years.

For example, if £100 is invested at 5% per annum compound interest, then after 1 year the interest earned is £5 (100 x 0.05) and the capital invested for the second year is £105. The interest earned in the second year is then £5.25 (£105 x 0.05) and this capital amount is carried forward to year 3.

Presenting this in tabular form:

Beginning of year	1	2	3	4	5
Capital	100	105	110.25	115.7625	121.550625
Interest during the year	100 x 0.05 = 5	105 x 0.05 = 5.25	110.25 x 0.05 = 5.5125	115.7625 x 0.05 = 5.788125	

Reproducing the table in algebraic form with ‘a’ representing capital and ‘i’ representing the rate of interest (in decimal form):

Beginning of year	1	2	3	4	5
Capital	a	$a + ai = a(1+i)$	$a(1+i) + a(1+i)i = a(1+i)^2$	$a(1+i)^2 + a(1+i)^2 i = a(1+i)^2(1+i) = a(1+i)^3$	$a(1+i)^3 + a(1+i)^3 i = a(1+i)^3(1+i) = a(1+i)^4$
Interest during the year	ai	$a(1+i)i$	$a(1+i)^2 i$	$a(1+i)^3 i$	

Therefore the value in year n is $a(1+i)^{n-1}$. Hence the example above with a = 100, i = 0.05, the value at the beginning of year 5 is:

$100 \times (1 + 0.05)^4 = 121.550625$

The sum of a geometric series is obtained easily by considering the series in its algebraic form (see table above):

$a, ar, ar^2, ar^3, \ldots, ar^{n-1}$

The sum of the first n terms is:

$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$

Multiplying both sides by r:

$rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^n$

(since $ar^{n-1} \times r = ar^{n-1+1} = ar^n$)

Subtracting rS_n from S_n gives:
$S_n – rS_n = a – ar^n$

Hence:
$S_n = (a – ar^n)/(1 – r) = a(1 – r^n)/(1 – r)$

Example 1:
Find the sum of the first 10 terms of the series
8, 4, 2, 1,…..

This is a geometric series since each term is obtained by multiplying the previous term by 0.5. Applying the formula above, the sum of the first 10 terms is:

$S_{10} = 8(1 – 0.5^{10})/(1 – 0.5) = 16 \times (1 – 0.00098) = 15.98$

Note the calculation of r^n yields a very small value of 0.00098 and as 'n' gets larger this value can only get smaller, e.g.

$0.5^5 = 0.03125$, $0.5^{15} = 0.000031$

Hence if the series has an infinite number of terms, r^n will be so small that in practise it will be zero. The formula for the sum of an infinite geometric progression is then:

$S_n = a(1 – 0)/(1 – r) = a/(1- r)$

TASK ONE

A financial analyst is analysing the prospects of a certain company. The company pays an annual dividend on its stock. A dividend of £5 has just been paid and the analyst estimates that the dividends will grow by 20% per year for the next five years, followed by annual growth of 10% per year for 5 years.

(a) Complete the following table:

Year	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Dividend										

(b) Then calculate the total dividend that will be paid for the next ten years.

TASK TWO

A stock begins to pay dividends with the first dividend, one year from now, expected to be £10. Each year the dividend is 10% larger than the previous year's dividend. In what year is the dividend paid larger than £100?

ACTIVITY TWO (Arithmetic Series)

TASK ONE

An economist believes that the size of a regional economy (as measured by GDP) can be accurately measured using an arithmetic progression.
He discovers that GDP over the past 5 years is as follows:

Year 1: $ 250 million
Year 2: $ 267.5 million
Year 3: $ 285 million
Year 4: $ 302.5 million
Year 5: $ 320 million

Assuming that the arithmetic progression is correct and a robust guide to the future, calculate the size of the regional economy in year 17 only, **using a formula**.

ANSWERS

ACTIVITY ONE

TASK ONE

(a)

Year	1	2	3	4	5	6	7	8	9	10
Dividend	£6.00	£7.20	£8.64	£10.37	£12.44	£13.69	£15.05	£16.56	£18.22	£20.04

(b) $S_n = a(1 - r^n)/(1 - r)$

The sum of the first 5 dividends is:

$S_5 = 6 \times (1 - 1.2^5)/(1 - 1.2) = 6 \times (1 - 2.488)/(-0.2) = £44.65$

The sum from year 6 to 10 is:

$S_{6\text{-}10} = 13.69 \times (1 - 1.1\hat{}^{5})/(1 - 1.1) = 13.69 \times (1 - 1.61)/(-0.1) = £83.55$

In total: 44.65 + 83.55 = £128.2.

TASK TWO

In order to answer this question we need to recall the expression for the value of a geometric series in a given period:

$T_n = ar^{n-1}$

Hence we need to find the 'n' that gives a T_n of £100 given an 'a' of £10 and an 'r' of 1.1. Hence:

$£100 = £10 \times 1.1^{n-1}$

$£100/£10 = £10 = 1.1^{n-1}$

In order to solve this problem we need to refer back to Guide One and the rules surrounding logs, namely:

$\log(a^n)=n\log(a)$

Therefore, taking the logs of both sides:

$\log(10) = (n-1)\log(1.1)$

$n-1 = \log(10)/\log(1.1) = 24.16$

Hence n = 24.16 + 1 = 25.16. Thus the dividend paid in year 26 will be greater than £100. In fact it will be £10 x 1.1^{26-1}= £108.35. The dividend in year 25 will then be £10 x 1.1^{25-1} = £98.50 (which is less than £100).

ACTIVITY TWO

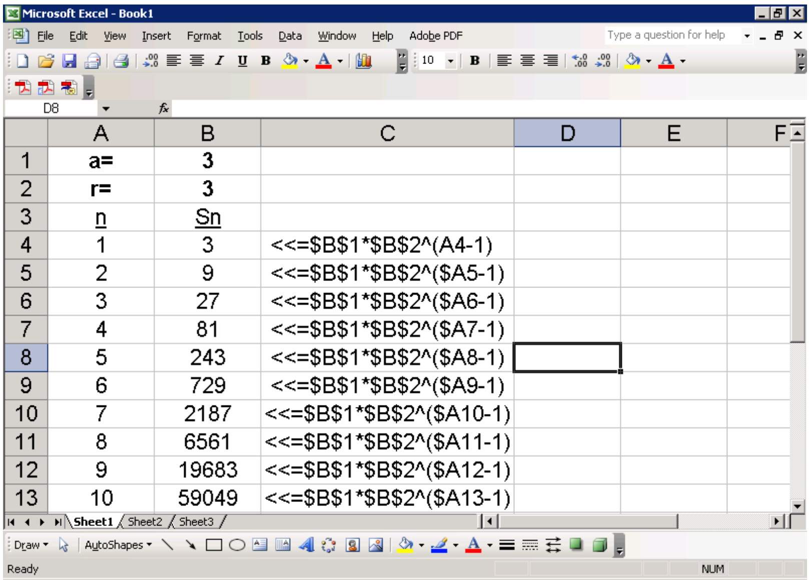
TASK ONE

$U_n = U_1 + (n-1) \times d$

U_{17} = \$250 million + (16 x \$17.5 million)
= \$250 million + (16 x \$17.5 million)
= \$250 million + \$280 million
= \$530 million

6. Top Tips

The use of Excel is integral to this topic. A quick and simple illustration of an arithmetic series is to enter 2 numbers in consecutive rows. If you now highlight the two cells and drag the mouse downwards you will see that Excel assumes an arithmetic series in generating the subsequent values. Similarly, Excel is very useful at illustrating geometric series. See the worksheet below as a simple example.



	A	B	C	D	E	F
1	**a=**	**3**				
2	**r=**	**3**				
3	n	Sn				
4	1	3	<<=$B$1*$B$2^(A4-1)			
5	2	9	<<=$B$1*$B$2^($A5-1)			
6	3	27	<<=$B$1*$B$2^($A6-1)			
7	4	81	<<=$B$1*$B$2^($A7-1)			
8	5	243	<<=$B$1*$B$2^($A8-1)			
9	6	729	<<=$B$1*$B$2^($A9-1)			
10	7	2187	<<=$B$1*$B$2^($A10-1)			
11	8	6561	<<=$B$1*$B$2^($A11-1)			
12	9	19683	<<=$B$1*$B$2^($A12-1)			
13	10	59049	<<=$B$1*$B$2^($A13-1)			

By varying the values for 'a' and 'r' you can generate a new geometric series.

7. Conclusion

Students should be encouraged to work thorough problem sets to help them develop their mathematical skills and confidence. This underpinning and knowledge and competency will be needed for subsequent material.

Section 3: Simple and Compound Interest

1. The concept of simple and compound Interest

As noted earlier, students will benefit from clear and cogent definitions at the beginning of the topic or module.

Definition of Simple Interest

Simple interest is a fixed percentage of the principal, P, that is paid to an investor each year, irrespective of the number of years the principal has been left on deposit. Consequently an amount of money invested at simple interest will increase in value by the same amount each year.

Algebraically, the amount of simple interest, I, earned on a deposit, P_0, invested at r% for t years is:

$$I = P_0 \times r \times t$$

Hence the total amount of money after t years is the principal plus the accrued interest:

$$P_0 + P_0 \times r \times t = P_0(1 + rt)$$

Definition of compound interest

The view above of only the principal earning interest is a very simplistic one and normally the interest on money borrowed is usually “compounded”. Compound interest pays interest on the principal plus on any interest accumulated in previous years.

The total value after t years when a principal, P_0, is compounded at r% per annum is:

$$P_t = P_0(1+r)^t$$

2. Presenting the concept of simple and compound Interest

Worked examples are a good way for students to understand and apply the concepts and mathematical techniques. They also provide a reference for students to return if they need to when they are working independently on problem sets. Two worked examples are provided below to help colleagues present these two concepts.

Presenting the calculation of simple interest

If you borrow £1000 for five years at a simple interest rate of 10% p.a., the amount of interest you pay is:

$$I = P_0 \times r \times t = £1000 \times 0.1 \times 5 = £500$$

Thus the cost of borrowing £1000 for five years at 10% p.a. simple interest is £500. The total amount due after five years would be:

$$P_0(1 + rt) = £1000 \times (1 + 0.1 \times 5) = £1500.$$

Presenting the calculation of compound interest

In order to demonstrate the fundamentals of compound interest consider a deposit of £1000 over 5 years at 10% per annum with interest compounded annually.

Principal at start of year	Interest paid each year	Total at end of year
£1000	£100	£1100
£1100	£110	£1210
£1210	£121	£1331
£1331	£133.10	£1464.10
£1464.10	£146.41	£1610.51

Note the £1610.51 could have been obtained directly as:

$$£1000 \times (1 + 0.1)^5 = £1000 \times 1.61051 = £1610.51$$

Repeating the above example but using algebraic notation rather than numbers:

Principal at start of year	**Interest earned each year**	**Total at end of year**
P_0	r x P_0	$P_0 + rP_0$ $= P_0(1 + r) = P_1$
P_1	r x P_1	$P_1+rP_1=P_0(1 + r)+ rP_0(1 + r) = P_0(1+r)(1+r) =$ $P_0(1+r)^2 = P_2$
P_2	r x P_2	$P_2+rP_2=P_0(1 + r)^2+ rP_0(1 + r)^2 = P_0(1+r)^2(1+r) =$ $P_0(1+r)^3 = P_3$
…	…	…
P_{t-1}	r x P_{t-1}	$P_0(1+r)^t = P_t$

Notice that the totals at the end of each year form a geometric progression (including the initial amount):

$$P_0,\ P_0(1+r),\ P_0(1+r)^2,\ P_0(1+r)^3,\ \ldots,\ P_0(1+r)^t$$

Hence as stated previously the total value after t years when a principal, P_0, is compounded at r% per annum is:

$$\mathbf{P_t = P_0(1+r)^t}$$

3. Delivering the concept of simple and compound Interest to small and larger groups

Students will need an initial and formal overview of simple and compound interest. This could be complemented by a more practical exercise where students visit banks and building societies – in person or online – to consider the different rates of interest offered by different financial products and to consider how the total return they would receive would be significantly affected by compounding.

Links to the online question bank

There are questions on investment appraisal including simple and compound interest at http://www.metalproject.co.uk/METAL/Resources/Question_bank/Economics%20applications/index.html

Video clips

The video clips for this Guide can be found at
http://www.metalproject.co.uk/Resources/Films/Mathematics_of_finance/index.html
Clips 3.01 to 3.03 (inclusive) offer good practical illustrations of compounding starting with an applied example of a student calculating how long it will take for them to save for a round the world trip.

4. Discussion

A simple competition and discussion could be created where students are paired and given a notional £1000 and a 10 year investment time frame and asked to select a savings account which would maximise their compounded interest. Higher ability students could look at foreign savings products and could factor in exchange rates when arriving at the Sterling figure.

5. Activities

Learning Objectives

LO1: Students learn the meaning of key financial terms- bond, coupon, interest, simple and compound interest

LO2: Students learn how to calculate simple and compound interest

LO3: Students learn the distinction between simple and compound interest

LO4: Students learn the impact which the frequency of compounding has upon the size of the total return

Task One

A bond's coupon is the annual interest rate paid on the issuer's borrowed money, generally paid out semi-annually. The coupon is always tied to a bond's face or **par value**, and is quoted as a percentage of par. For instance, a bond with a par value of £1,000 and an annual interest rate of 4.5% has a coupon rate of 4.5% (£45).

Say you invest in a six-year bond paying 5% per year, annually. Assuming you hold the bond to maturity, you will receive 6 interest payments of £250 each, or a total of £1,500. Plus the par value of £1000.This coupon payment is simple interest.

You can do two things with that simple interest—spend it or reinvest it. Determine the total amount of money you will have after six years, assuming you can reinvest the interest at 5% per annum.

Task Two

So far, it has been assumed that compound interest is compounded once a year. In reality interest may be compounded several times a year, e.g. daily, weekly, monthly, quarterly, semi-annually or even continuously.

The value of an investment at the end of m compounding periods is:

$$P_t = P_0[1 + {}^{r}/_{m}]^{m \times t}$$

Where m is the number of compounding periods per year and t is the number of years.

Using this information, solve the following problem:

(a) £1,000 is invested for three years at 6% per annum compounded semi-annually. Calculate the total return after three years.

(b) What would the answer be if the interest was compounded annually?

(c) If the interest was compounded monthly is it true that the total amount after three years would be less than £1195.00?

(d) Using your answers to (a) – (c) what can you infer about the frequency of compounding and the size of the total return?

Task Three (Using the formula $P_t = P_0e^{rt}$)

It follows that when we compound interest continuously the value of the investment at the end of the period becomes $P_t = P_0e^{rt}$.

Worked example

A financial consultant advises you to invest £1,000 at 6% continuously compounded for three years. Find the total value of your investment.

$$P_t = P_0e^{rt} = £1{,}000 \times e^{(0.06 \times 3)} = £1{,}000 \times 2.7183^{0.18} = £1{,}197.22$$

In Excel to raise 'e' to the power of another number you use the "EXP" function.

(a) Use Excel to set up a table comparing the growth of £1 invested for 25 years at 20% assuming interest is compounded (i) annually; (ii) quarterly; (iii) monthly and (iv) continuously.

(b) Graph the outcome.

(c) What conclusion can be drawn regarding the frequency of compounding?

ANSWERS

Task One

Time (years)	Cash Flow received	Interest Earned at 5%
1	£250	£69.07
2	£250	£53.88
3	£250	£39.41
4	£250	£25.63
5	£250	£12.50
6	£1,250	£0.00
Totals	**£2,500**	**£200.48**

When you reinvest a coupon, however, you allow the interest to earn interest. The precise term is "interest-on-interest," (i.e. compounding). Assuming you reinvest the interest at the same 5% rate and add this to the £1,500 you made, you would earn a cumulative total of £2,700.48, or an extra £200.48 (of course, if the interest rate at which you reinvest your coupons is higher or lower, your total returns will be more or less).

Task Two

(a) $P_3 = £1{,}000 \times (1 + {}^{0.06}/_{2})^{2 \times 3} = £1{,}000 \times (1.03)^6 = £1194.05$

(b)

Time (years)	Principal	Interest Earned	Total at End of Period
1	£1,000.00	£60.00	£1,060.00
2	£1,060.00	£63.60	£1,123.60
3	£1,123.60	£67.42	£1,191.02

Which is the same as $£1{,}000 \times (1+0.06)^3 = £1{,}191.02$.

(c) **FALSE** since we would get

Time (months)	Principal	Interest Earned	Total at End of Period
1	£1,000.00	£5.00	£1,005.00
2	£1,005.00	£5.03	£1,010.03

3	£1,010.03	£5.05	£1,015.08
..	..	..	..
..	..	..	..
35	£1,184.83	£5.92	£1,190.75
36	£1,190.75	£5.95	£1,196.70

or £1,000 x $(1 + {}^{0.06}/_{12})^{12 \times 3}$ = £1,000 x $(1.005)^{36}$ = £1,196.68 (note difference is due to rounding errors)

(d) Hence the greater the compounding frequency the greater the total return. Thus if we compound daily the total return would be:

£1,000 x $(1 + {}^{0.06}/_{365})^{365 \times 3}$ = £1,197.20.

To illustrate what happens as the compounding frequency is increased, consider the table below.

Compounding Frequency	Total at End of Period
1	£1,191.02
2	£1,194.05
4	£1,195.62
8	£1,196.41
12	£1,196.68
52	£1,197.09
365	£1,197.20
730	£1,197.21
1460	£1,197.21
5000	£1,197.22
10000	£1,197.22
50000	£1,197.22

As the value of m (the compounding frequency) increases the value of the investment becomes larger, but never exceeds £1.197.22

Note the final value is arrived at from:

$$£1,000 \text{ x } (1 + {}^{0.06}/_{50000})^{50000 \text{ x } 3} = £1,197.22.$$

Note: Higher ability students might want to consider the following explanation:
By allowing m to approach infinity interest is being added to the investment more and more frequently and can be regarded as being added continuously, such that:

$$\lim_{m\to\infty} £1000x\left(1+\frac{0.06}{m}\right)^{mx3} = £1,197.22$$

Here we applied this formula:

$$P_t = P_0[1 + {}^r/_m]^{m \text{ x } t}$$

with P_0 set at £1,000, ‘r’ set at 0.06 and ‘t’ set at 3 and varying m. If we now set P_0 at £1, ‘r’ at 100% (i.e. 1) and ‘t’ set at 1 year we arrive at the following answer for ‘m’ set at 50,000:

$$P_t = £1\text{x } [1 + {}^1/_{50,000}]^{50,000} = 2.7183$$

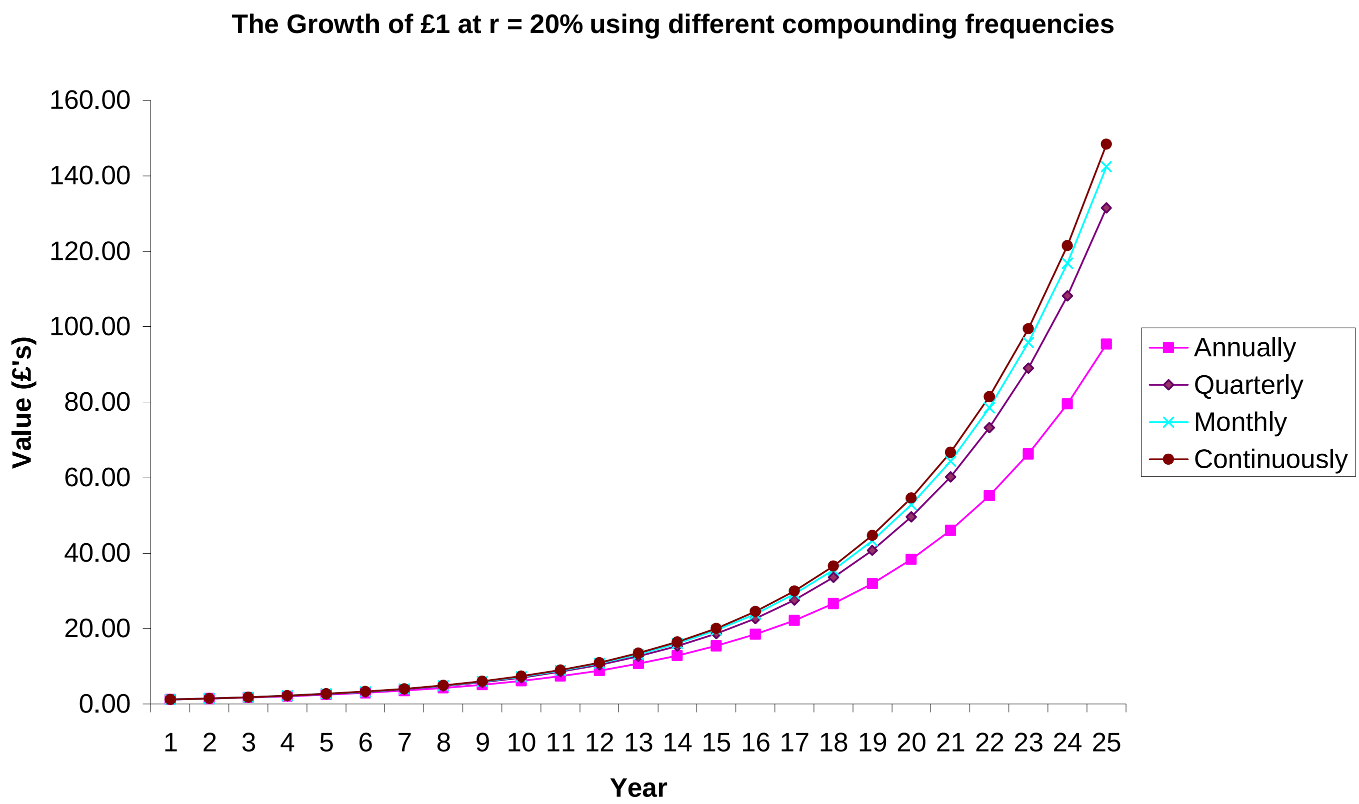
Thus we can say that:

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m} = 2.7183$$

You may recognise this number, 2.7183, the natural logarithm, e. Note log to the base e of 2.7183 (denoted $\log_e(2.7183) = 1$).

Task Three

(a) and (b)



(c) We can observe from the chart that the continuous method of compounding leads to the greatest cumulative total after 20 years, while the annual method gives the smallest sum. Furthermore, the longer the investment is left on deposit the wider the differences when compounded by the different methods.

6. Top Tips

Remind students that the interest rate is always quoted as a percentage, i.e. r%. Many students erroneously enter interest rates into a variety of financial problems. As a good check it is useful to ask students to undertake a simple 'commonsense' check: look at the answer and ask if its "reasonable". For example if I invest £100 for one year at 10% would I really expect to get £1100 back at the end of the year (£100 x (1+10)) or is £110 more realistic (£100 x (1+0.01))? This intuition is impossible to teach but students can acquire and develop it if they practise stopping and reflecting on their answers rather than accepting whatever appears on their calculator screens.

Students appreciate real examples rather than "made up" examples. You will find a huge wealth of examples at http://www.nsandi.com/products/frsb/calculator.jsp

Section 4: Investment Appraisal

1. The concept of investment appraisal

It would be helpful to start by explaining what is meant by the term, "Investment Appraisal" and also that it is necessary to first consider the time value of money and the concept of present value.

2. Presenting the concept of investment appraisal

A good way to get students thinking about investment appraisal is to ask the group of students "would you rather have £100 now or £100 in a years time?". One would expect everyone to respond with "£100 now!". Then repeat the question at £85, £90, £95, £96, £97, £98 and £99. One would be surprised if anyone said yes at £85, £90 or even £95.

But if they do you can question their logic. They may choose to take the money now because they favour present consumption over future consumption. But if they argue that they could the

£85, £90 or £95 now and invest it to yield more than £100's in a year's time then we can focus on the interest rate that their investment will earn.

This can be developed by asking students to think about the present value of £100 and then you can then introduce the real world of tax on interest income at 20%. And again ask the question of how much they would need now to be indifferent to £100 in a year's time. If students are comfortable with this concept you could also introduce the issue of default risk and the idea that investors may demand a risk premium. At a higher level you would do this by comparing some sovereign debt with some corporate debt that are identical in all respect, other than the issuer. The corporate debt would be priced cheaper to reflect the risk premium.

3. Delivering the concept of investment appraisal to small and larger groups

Students need help understanding the practical consequences of 'time value'. A concise overview with opportunities to practise applying these concepts will be valuable.

For example, lecturers could ask students to assume that only one interest rate existed, r, and that all individuals could borrow and lend at then the present value of £100 is:

$$\frac{£100}{1+r}$$

e.g. r = 10%, £100/1.1 = £90.91.

Alternatively:

PV = discount factor x Cash Flow

This discount factor is the value today of £1 received in the future and is usually expressed as the reciprocal of 1 plus a rate of return:

$$\text{Discount Factor} = \frac{1}{1+r}$$

Note as r gets bigger the denominator will rise and hence the whole term, the discount factor will fall. Hence the PV of a future cash flow falls as the interest rate rises. The table below shows the present value of £100 received in a year's time as the interest rate rises:

Interest Rate	PV of £100 received in one year
1%	£99.01
2%	£98.04
3%	£97.09
4%	£96.15
5%	£95.24
6%	£94.34
7%	£93.46
8%	£92.59
9%	£91.74
10%	£90.91
11%	£90.09
12%	£89.29
13%	£88.50
14%	£87.72
15%	£86.96
16%	£86.21
17%	£85.47
18%	£84.75
19%	£84.03
20%	£83.33

If we were considering the present value of £100 received in two, three, four or 'n' years then it would be:

$$\frac{£100}{(1+r)^2}, \frac{£100}{(1+r)^3}, \frac{£100}{(1+r)^4}, \ldots, \frac{£100}{(1+r)^n}$$

The table below assumes an interest rate of 10% and shows what happens to the present value of £100 as it is received further away into the future.

Year (n)	PV of £100 received in year n
1	£90.91
2	£82.64
3	£75.13
4	£68.30
5	£62.09
6	£56.45
7	£51.32
8	£46.65
9	£42.41
10	£38.55
11	£35.05
12	£31.86
13	£28.97
14	£26.33
15	£23.94
16	£21.76
17	£19.78
18	£17.99
19	£16.35
20	£14.86

This the present value of some future cash flow, C, received in ‘n’ years time when the interest rate is r % p.a. is:

$$PV = \frac{C}{(1+r)^n}$$

The Net Present Value Rule

Students will need to be clear of the “The Net Present Value rule“ with regards to investment appraisal is:

Accept any project if its NPV > 0 or if NPV=0
Reject a project of its NPV < 0

Suppose a project has a positive NPV, but the NPV is small, say, only a few hundred pounds then the firm should still undertake that project if there are no alternative projects with higher NPV as a firms wealth is increased every time it undertakes a positive NPV project. A small NPV, as long as it is positive, is net of all input costs and financing costs so even if the NPV is low it still provides additional returns. A firm that rejects a positive NPV project is rejecting wealth!

The Internal Rate of Return

Students can be guided to this last concept having worked through the earlier material on PV and NPV. It is important for them to be clear exactly what IRR means, namely: the Internal rate of return of a project can be defined as the rate of discount which, when applied to the projects cash flows, produces a zero NPV. That is, the IRR decision rule is then: “invest in any project which has an IRR greater than or equal to some predetermined cost of capital”.

Links to the online question bank

There are questions on investment appraisal at http://www.metalproject.co.uk/METAL/Resources/Question_bank/Economics%20applications/index.html

Video clips

There are three very clear and ‘applied’ clips at http://www.metalproject.co.uk/Resources/Films/Mathematics_of_finance/index.html. Clips 3.05 to 3.07 (inclusive) would be useful for delivering material on PV, NPV and discounted cash flow.

4. Discussion

There is wide range of issues that can be discussed and the video clips could be a useful starting point. Students could be asked to find a large investment project and to prepare a brief presentation on how investment appraisal techniques could have been employed. For example, the Government's funding of the new Wembley Stadium would have involved looking at a variety of different schemes and with different costs and assumptions. How might the Government have selected the chosen project? What factors might Government economists have taken into account when advising Ministers? This discussion could move into wider issues regarding the quality of forecasts, projections and the risks associated with large and long-term investments.

5. Activities

ACTIVITY ONE

<u>Learning Objectives</u>

LO1: Students to learn how to calculate present values

LO2: Students learn how to apply their understanding of present values to solve annuity problems

LO3: Students learn how to use formulae to solve financial problems

LO4: Students learn how to use Excel to answer financial questions

TASK ONE (Present Values and Annuities)

Suppose you win a £1m lottery prize and are offered the choice between taking the whole £1m now or £50,000 per year for 25 years. Which would you choose?

TASK TWO

Follow the worked example below and then attempt the task below (Se TASK)

Worked Example

The financial advisers Alexander Forbes quote annuity rates on there website:
http://www.annuity-bureau.co.uk/Annuity+Rates/Current+annuity+rates/

You can either choose an annuity with a fixed return or an annuity that increases in line with the RPI. A selection of quotes is presented below:

Escalation: Level

Last updated: 20 November 2006

Provider	Rank	Income	Provider	Rank	Income
Male 55, Single Life			Female 55, Single Life		
Scottish Equitable	1	£5,718.00	Norwich Union	1	£5,580.96
Canada Life	2	£5,713.92	Canada Life	2	£5,495.04
Prudential	5	£5,598.84	Prudential	5	£5,307.24
Friends Provident	7	£5,230.20	AXA Sun Life	7	£4,896.00

Source: http://www.annuity-bureau.co.uk/AF_CMS/Resources/Annuity%20bureau/rates_level.html (accessed 13/11/2006)

The derivation of the value of the figure of £5,718 from Scottish Equitable can be illustrated using the concept of Present Value. The income values given in the table quote how much £100,000 will purchase per annum for the rest of your life. Recall that to find the present value of an annuity we need to know the values for r and t, where t in this context will represent life expectancy.

The national statistics office (http://www.statistics.gov.uk/) maintains a database of life expectancy according to age and gender. These can be found by searching for “Interim life tables” at the above website. The interim life table indicates that the average life expectancy of a male aged 55 living in the United Kingdom would be 24.67 years (based on data for the years 2003-2005). For a woman it would be 28.04 years. Note this refers to the average number of years an individual will survive after the age of 55 years.

Source: http://www.statistics.gov.uk/downloads/theme_population/Interim_Life/ILTUK0305.xls

If interest rates were 5% then the PV of £1 received for the next 24.67 years would be:

$$\text{PV of annuity} = C \times \left[\frac{1}{r} - \frac{1}{r(1+r)^t}\right] = £1 \times \left[\frac{1}{0.05} - \frac{1}{0.05(1+0.05)^{24.67}}\right] = £13.9981$$

So if £100,000 was available to buy an annuity the annuity would be quoted as:

£100,000/£13.9981 = £7143.84

Note above the annuity from Scottish Life is quoted as £5,718. Using “Goal Seek in Excel” we can find the interest rate used by Scottish Life in their calculations.

C=	1
r=	2.88%
T=	24.67
PV=	17.4886
Annuity Rate=	£5,718.00

Or,

$$\text{PV of annuity} = C \times \left[\frac{1}{r} - \frac{1}{r(1+r)^t}\right] = £1 \times \left[\frac{1}{0.0288} - \frac{1}{0.0288(1+0.0288)^{24.67}}\right] = £17.4886$$

Hence we find that they used the rate of 2.88% (assuming the same life expectancy inputs).

TASK

Given the life expectancy values above determine the discount rates used by Canada Life in valuing their annuities.

ACTIVITY TWO

Learning Objectives

LO1: Students to learn how to calculate net present values and apply the IRR rule

LO2: Students to learn how to make independent evaluations of investment projects using NPV and IRR methodology

Task One (NPV)

Consider three alternative projects, A, B and C. They all cost £1,000,000 to set up but project’s A and C return £800,000 per year for two years starting one year from set up. Projects B also returns £800,000 per year for two years, but the cash flows begin two years after set up. Whilst project C costs £1,000,000 to set up it initially requires £500,000 and £500,000 at termination (a clean-up cost for example).

If the interest rate is 20% which is the better project?

Task Two (IRR)

The Internal rate of return of a project can be defined as the rate of discount which, when applied to the projects cash flows, produces a zero NPV. That is, the IRR decision rule is then:

"invest in any project which has an IRR greater than or equal to some predetermined cost of capital".

Consider a project that requires £4,000 investment and generates £2,000 and £4,000 in cash flows for two years, respectively. What is the IRR on this investment?

ANSWERS

ACTIVITY ONE

Task One

If the interest rate were zero, the 25 payments of £50,000 would be chosen as this amounts to 25 x £50,000 = £1,250,000. However, interest rates are generally not zero and the present value of the £50,000 received in 10, 15 and 25 years time will be greatly reduced.

In order to determine the present value of the cash flows an appropriate interest rate needs to be determined. Lets assume an interest rate of 5%.

A regular payment over a fixed period of time is referred to as an annuity.

$$\text{PV of annuity} = C \times \left[\frac{1}{r} - \frac{1}{r(1+r)^t} \right]$$

Present value of the regular cash flow is £704,700. Hence the lottery winner should accept the £1m now.

TASK TWO

	Male	**Female**
C=	1	1
r=	2.87%	3.26%
T=	24.67	28.04
PV=	17.5011	18.1982
Annuity Rate=	£5,713.92	£5,495.04

ACTIVITY TWO

Task One

Note that the net cash flow for all three projects, ignoring the time value of money, is -£1,000,000 + £1,600,000 = +£600,000.

However, when the time value of money is taken into account one project may be preferable to the others. Without doing any calculations can you determine the order of preference?

Consider A versus B. They both cost the same but B's cash flow returns occur later than A's. Hence A is preferable to B.

Consider A versus C. They both have the same time pattern and size of returns and both cost the same to set up. However the payout to establish C is split with some cashflow up front and some at the end. Hence C is preferable to A.

Hence the rank is C, A, B.

However in may cases, the method of comparison is more complicated. In such cases, NPV analysis must be applied:

Project A

interest rate	**20%**	<<= you can change this			
Year	**0**	**1**	**2**	**3**	**4**
Cash Flow	-$1,000,000	$800,000	$800,000	$0	$0
discount factor	1.000	0.833	0.694	0.579	0.482
PV	-$1,000,000.00	$666,666.67	$555,555.56	$0.00	$0.00
NPV=	**$222,222.22**	Rank=	2		

Project B

interest rate	**20%**				
Year	**0**	**1**	**2**	**3**	**4**
Cash Flow	-$1,000,000	$0	$0	$800,000	$800,000
discount factor	1.000	0.833	0.694	0.579	0.482
PV	-$1,000,000.00	$0.00	$0.00	$462,962.96	$385,802.47
NPV=	**-$151,234.57**	Rank=	3		

Project C

interest rate	**20%**				
Year	**0**	**1**	**2**	**3**	**4**
Cash Flow	-$500,000	$800,000	$300,000	$0	$0
discount factor	1.000	0.833	0.694	0.579	0.482
PV	-$500,000.00	$666,666.67	$208,333.33	$0.00	$0.00
NPV=	**$375,000.00**	Rank=	1		

This is consistent with our previous intuitive analysis. These calculations are available at www.lawseconomics.co.uk/metal/npvandr.xls You will notice that if you lower the interest rate in cell B1 that the NPV rises and if you raise the interest rate the NPV falls.

Hence r↑, NPV↓ and r↓, NPV↑

Task Two

In order to verify the figure of 28.08% from above we can either find it by hand, using one of two ways, or use "Goal Seek" or the IRR function in Excel.

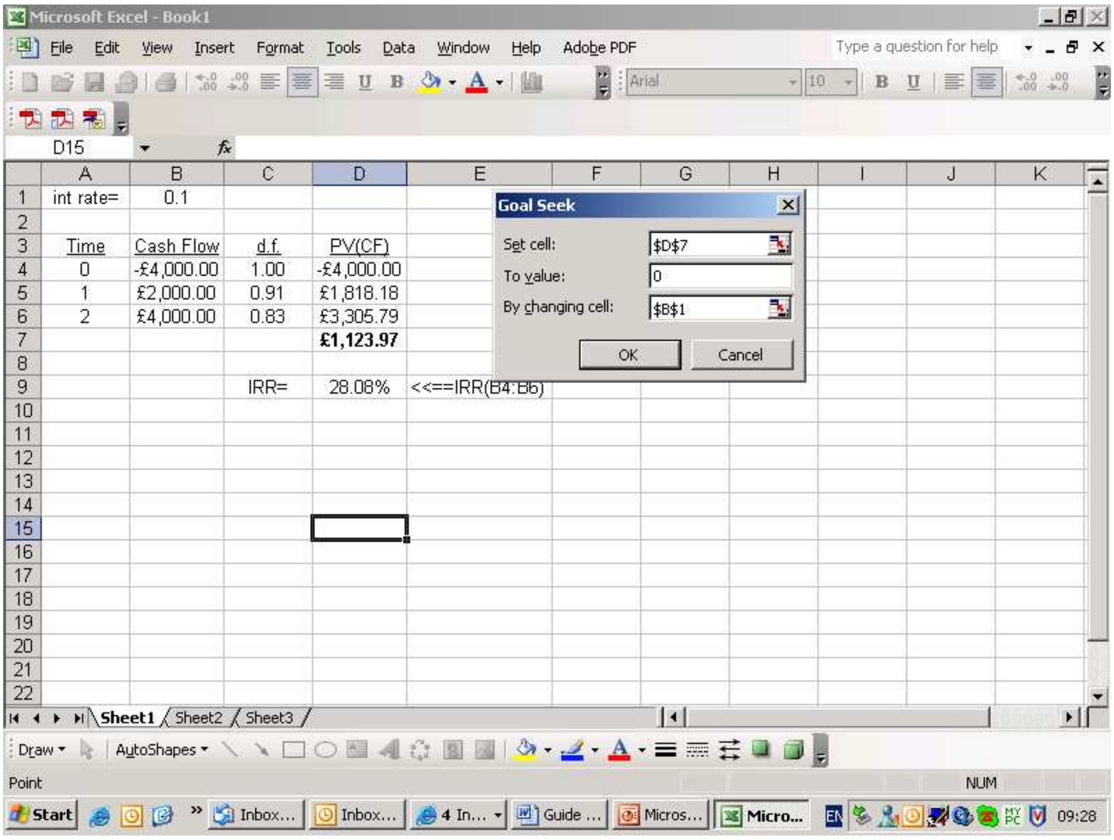
First of all using the IRR function. Note here we simply select the function and highlight the cash flows. We do not highlight the Present Values of the cash flows.

int rate=	0.1

Time	Cash Flow	d.f.	PV(CF)
0	-£4,000.00	1.00	-£4,000.00
1	£2,000.00	0.91	£1,818.18
2	£4,000.00	0.83	£3,305.79
			£1,123.97

IRR=	**28.08%**

Now using "Goal Seek" we arrive at the same answer.

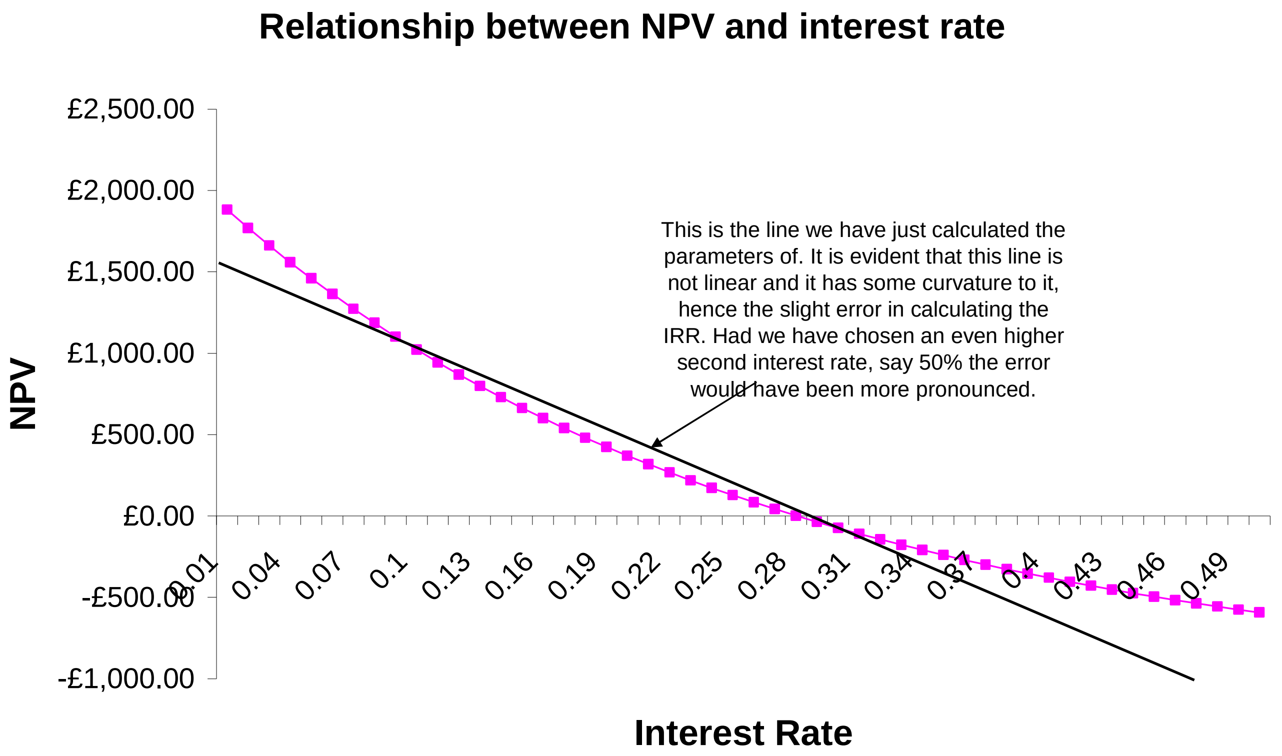


int rate= 0.2808

Time	Cash Flow	d.f.	PV(CF)
0	- £4,000.00	1.00	- £4,000.00
1	£2,000.00	0.78	£1,561.55
2	£4,000.00	0.61	£2,438.45
			£0.00

IRR= 28.08%

The graph below helps to illustrate the link between the interest rate and the NPV:



6. Top Tips

Annuities can be explored and the enormous number of permutations that can be used to illustrate the concepts can be found at www.annuity-bureau.co.uk.

7. Conclusion

The success of this summative material will turn crucially upon students having acquired and developed the understanding and mathematical skills introduced at the beginning. Give students lots of opportunities to research ‘real-world’ financial products and to share their findings and independent application as they progress through this unit or module.