METAL Teaching and Learning

Guide 4: Linear Programming

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Section 1

Introduction to the guide

1. This guide intends to serve as a useful resource for colleagues delivering linear programming to undergraduate students. Students are assumed to have a basic grasp of mathematics but there is no presumption that they have knowledge or any practical understanding of linear programming.
2. There are three main threads to the guide. The first thread explores how colleagues might deliver the conceptual or theoretical principles of linear programming. The second provides some pointers on how students could acquire a sound grasp of the mathematical concepts and to see how they can be applied to solve economic problems as well as then being able to apply this technique with confidence. The last focuses on encouraging and supporting students to make logical, sensible and cogent inferences of linear programming results including some critical evaluation of their findings.
3. All economists need to make sense of their findings and be able to show how their results can help solve an economic problem or issue. It is therefore helpful for colleagues to reflect on ways in which linear programming can be contextualised to help bring it alive and for students to realise it is a simple but effective tool rather than something which is abstract or of limited use.
4. Colleagues will be aware that very few economics undergraduates will have encountered linear programming before although a minority of students might have used this tool through applied business courses e.g. operational research, business mathematics or some higher statistics courses. Given the wide range of backgrounds and levels of prior attainment, it will be important for lecturers to consider ways to differentiate linear programming material.
5. ‘Linear programming’ is perhaps an uninviting title and students might well have misconceptions about what it actually means. In this author’s experience, students have initially suggested that linear programming is simply a synonym for computer programming or geometry, a type of ‘hard maths’ or that it involves merely graphing functions. This is perhaps unsurprising given that the label is quite opaque.
6. Lecturers have a clear interest and responsibility to begin with a very clear and crisp introduction to describe and explain exactly what linear programming means, the basic methods and how it can be used.
7. A large part of this guide then is designed to offer practical strategies to help students learn

more about economics and economic problems by improving their mathematical abilities and specifically linear programming. We can expect students to learn best when they have excellent teaching and student confidence and self-esteem will be obvious but sometimes elusive ingredients in this process.

Section 2: Linear Programming

1. The Concept of Linear Programming

8. The concept of linear programming is simple: the mathematical method of trying to achieve something whilst taking into account a number of constraints. Typical examples include:

   - maximising consumer satisfaction or utility subject to a budget constraint;
   - maximising profits subject to cost constraints;
   - minimising a firm's wage bill taking into account a given capital-labour ratio; or
   - maximising revenues for a sales maximising firm subject to the own price elasticity of demand.

9. Put more formally, **linear programming** problems are really about finding optimal solutions to problems which are expressed in terms of an entity which needs to be optimised (also known as 'the objective function') given certain constraints. Both the objective function and all of the constraints are expressed as linear equations eg. $y = 4x+7$ etc.

2. Presenting the concept of linear programming

10. An effective way to present the concept of simple linear programming is to start with simple 'real world' examples. For example, students could be first introduced to the two core concepts of 'objective functions' and 'constraints'.

(a) Understanding and Contextualising Objective Functions

11. A simple way to contextualise objective functions is to give real-world examples. For example, lecturers might want to prepare the whiteboard or Powerpoint presentation with a range of economic agents and to show a range of possible objectives or goals. The lecturer could ask students to match the agents to the objectives.

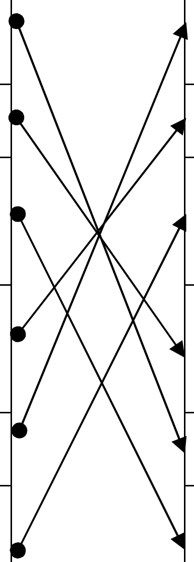
12. For example,

Economic agent
HM Treasury
An airline
A consumer
A large multinational car manufacturer
The Department of Health
A university economics department

Objective Function
Increase awareness, knowledge and understanding of sexually transmitted disease
Maximise sales of a new model
Maximise value-added achieved by undergraduates
Minimise fuels costs
Maximise tax revenue
To maximise satisfaction for a given amount of household income

Would yield the following mapping:

Economic agent
HM Treasury
An airline
A consumer
A large multinational car manufacturer
The Department of Health
A university economics department



Objective Function
Increase awareness, knowledge and understanding of sexually transmitted disease
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13. Students could be given an opportunity to think of their own objective functions. For

example, what are their personal targets or goals? This could be linked with other units of work eg. to discuss whether employees are wage maximisers or simply satisficers. Higher ability students could be encouraged to reflect to what extent objective functions are easy to frame in practice. For example, can we really assume that all firms are profit maximisers and even if we could is it always straightforward to express what the profit function is, particularly in the case of large multi-product conglomerate.

14. At this preliminary stage, lecturers might want to look at the video clip on Linear Programming introduction (Field 4.A.1). This reinforces the idea of objective functions but then prepares students for the next part of the analysis: economic constraints.

(b) Understanding and Contextualising Constraints

15. In the same way, lecturers could introduce the notion of constraints in a descriptive but engaging way. It might be useful to go back to the original table of economic agents and objective functions. Students could select one or more economic agents and consider the typical constraints or limiting factors that each selected agent would face.

16. For example, in the case of a consumer, lecturers might wish to share the following as an example:

Consumer **Typical Objective Function**	**Consumers face these typical constraints**
Maximise consumer satisfaction or utility through the consumption of goods and services.	- A fixed or given income - Prices of goods - A fixed quantity of goods available to them - Some desired goods might be unavailable or prohibited eg. alcohol to under-18s

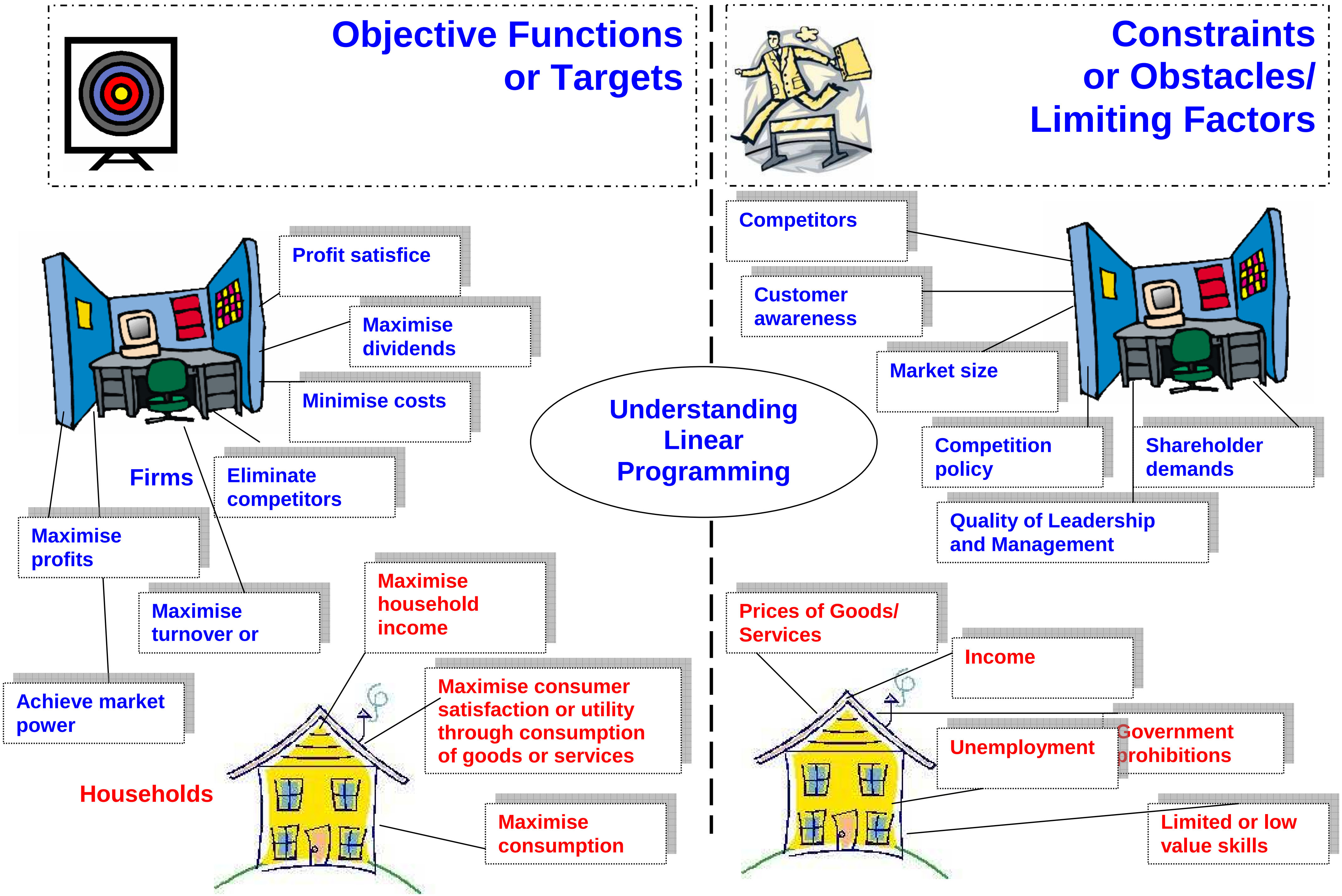
17. This could be developed to create a whole ‘economics scheme’ where economic parties or agents are considered and typical objective functions and constraints are identified and

shared. A simple example is provided on the next page together with a simple summary of objectives and constraints for firms and households. The creation of a multimedia presentation such as this could be particularly effective with small groups. It could be used for larger groups too with students sending their work electronically to the lecturer to be projected or displayed as a starter activity for the next seminar.

18. Clearly, opportunities exist for other methods of summarising and recording including mindmapping, spider-diagrams or simple tables. This could be extended to include objective functions and constraints for shareholders, employees /managers, Government, interest groups such as environmentalists etc. This could be used to create a plenary session on basic macroeconomics e.g. the agents in an open macroeconomy with government. A useful and free mindmapping software tool (FreeMind) can be downloaded from http://freemind.sourceforge.net/wiki/index.php/Main_Page#Download_and_install

19. This work could support the video clip (Studio 4.A.1) where a full LP problem is articulated. Students will have already thought about ‘real world’ LPs and the video clip then reinforces this with a really good example of Belgian Chocolates.

Understanding Objective Functions and Constraints: A Simple Overview of Households and Firms



3. Delivering the concept of linear programming to small or larger groups

20. After a descriptive introduction, all students will need to acquire, develop and apply a sound understanding of linear programming. In this Guide, much of what we discuss is limited to purely graphical solutions i.e. it does not cover 'high level' linear programming or many issues related to non-linear functions, problems involving complex numbers or problems which require the use of Lagrangeans etc.

(a) Delivering the concept of the feasible region optimisation to larger groups

21. Larger groups might find a single summary slide helpful to show that a linear programming problem is typically about maximising or minimising something. A maximisation problem is usually easier for students to understand – maximising sales or profits or utility – with the optimisation point being shown as the greatest amount that could be yielded given the constraints given a feasible region.

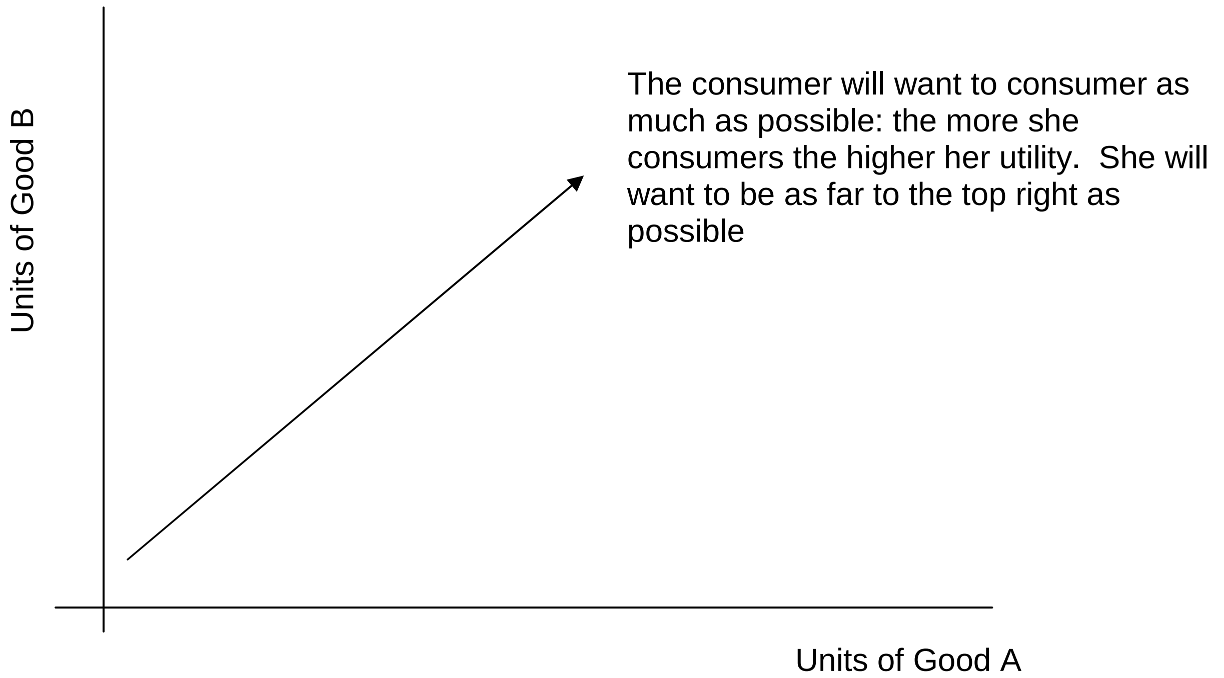
22. In the example, below a simple descriptive linear programming problem is solved first by working out the feasible region and then adding the optimisation point. This reinforces the technique outlined in the video clip Studio 4.A.1

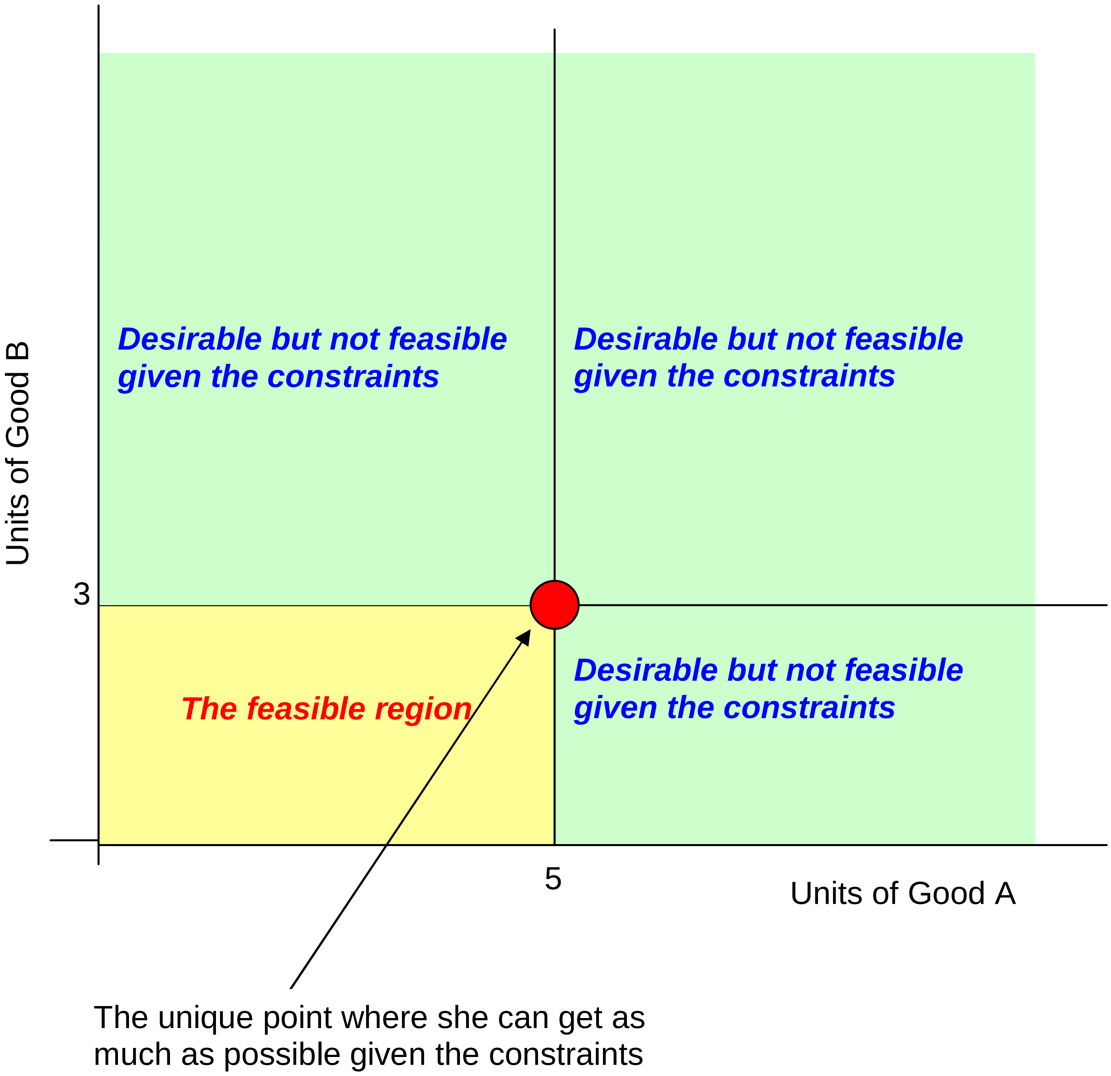
Example: Maximising income

A consumer receives equal utility (U) from two products (A) and (B)

She can consumer a maximum of 5 units of A and a maximum of 3 units of B.

Therefore, we can show





(b) Delivering the concept of the feasible region to smaller groups

23. Smaller groups might find it helpful to create simple feasible areas and optimal points using large and laminated graph paper and using cardboard strips to represent constraints. This also links with the video clip studio 4.A.1.

24. Lecturers could provide simple examples to improve students' confidence in identifying feasible regions and optimal points. This could be further extended by asking small groups to write their own simple linear program problems and sharing them with other groups. Students could be encouraged to share written LP problems with other groups and to test if they can formulate the LP. There might well be alternative ways of formulating a LP for a given scenario and the discussion and feedback which this could encourage might be very

good for students' learning. Some possible activities are set out in Section 5 below.

(c) Delivering the concept of non-linear programming

25. Much of this Guide focuses on linear programming. There is potential for higher ability students to extend their learning with non-linear programming and this is supported by the video clips. The video introduction to non-linear functions (Field 4.B.1) would be a good starting point.

26. Large groups could be shown a simple Powerpoint presentation which posits some economic problems but which do not follow simple linear functions. For example,

    - minimising costs where costs are not linear eg. linking with concepts of economies fo scale and scope;

    - maximising profits where profit functions are rarely linear since the equilibrium price will tend to fall as more of a product is produced assuming ceteris paribus; and

    - maximising revenue subject to the constraints of price elasticity of demand i.e. where total revenue will tend to rise where the price elasticity of demand is less than (modulus) 1.

27. This larger group work could be complemented by the video clip on non-linear supply and demand curves in the labour market (Field 4.B.3).

28. Students in smaller groups could be encouraged to undertake their own independent research on likely non LPs and to summarise their thoughts and findings in a simple presentation.

4. Discussion questions

Discussion question 1

Students could discuss ways in which linear programming could be used to help solve contemporary economic problems. The video clip gives some really good examples – the Belgian chocolate company for instance – and students could think about other variables which need to be maximised or minimised.

For example, what is the Government's objective function? What is it that we believe the

Government is trying to maximise? Is it merely votes or is it something wider and more nebulous? Could the objective function be defined as 'social welfare' and if so, what exactly does this term mean?

This could prompt some high level discussion about not just the application of linear programming but more fundamental questions about the goals or objectives of key economic agents.

Discussion question 2

Students could be asked to consider situations where relationships might not follow a simple linear pattern. Some students with greater economic knowledge and understanding might make links to costs and ideas of economies of scale or scope. Other links might include concepts of diminishing marginal returns to a factor or of utility. This could be complemented by an exploration of what such relationships or functions might look like eg. perhaps through a simple sketching activity.

Discussion Question 3

Students could be asked to complete a table summarising their discussions on how LP could be used in industry. A template is provided below together with some possible ideas.

Industry/ Business	How could LP be used?
Airline Industry	Minimising the amount of time an aeroplane is on the ground subject to constraints on handling time.
Car factory	Maximising the profit of a factory by choosing the optimal combination or mix of vehicles to produce subject to constraints on labour and capital.

A hospital	Maximising the number of operations which are undertaken each day subject to constraints on the number of surgeons available and their hours of work.
A football club	Maximising the revenue for a football match subject to capacity constraint of the ground and the different types or 'niche' of football supporter eg. terrace, box, executive

5. Activities

ACTIVITY ONE

<u>Learning Objectives</u>

LO1. Students to construct simple objective functions and constraints.

LO2. Students to identify simple feasible regions and optimal points.

Students are put into small groups and given large and laminated graph paper.

The axes are unlabelled and unscaled and the lamination allows students to 'write and wipe'.

Students are given thin cardboard strips to represent linear constraints.

Task One

In your small groups read the following passage and answer the questions which follow

"MFG is small car manufacturer producing two models of handmade cars. These cars are prestige vehicles and command a very high price. The two models are well known throughout the specialist car market as X and Y. Both car X and car Y are very profitable. Car X produces £2,000 of profit for each vehicle produced whilst Car Y produces £3,000. the firm is motivated purely by profit.

The company cannot produce as many cars as it would like. It cannot produce more than 15 cars in a year because they all have to be hand finished. For operational reasons, there are also limitations on the combinations of cars X and Y which can be manufactured in a given year. It is not possible given the current technology for more than 3 model Y cars to be produced for each model X car. This can create a headache for MFG."

(a) Write in words MFG's objective function

(b) In your group, translate the objective function into a simple expression using X to show the number of model X and Y to denote the number of model Y vehicles that can be manufactured in a year.

(c) How many constraints are there to this problem? Work out what the expressions are for these constraints.

(d) Using your laminated paper and strips, construct the linear program. How many of each should be produced?

ANSWERS

(a) A descriptive objective function could be, "To maximise profit from the production of vehicle X and Y"

(b) This objective function could be articulated as:

Let P = profit from the production and sale of X and Y cars in a year.

Let X = the number of vehicle X produced and sold in a year.

Let Y = the number of vehicle Y produced and sold in a year.

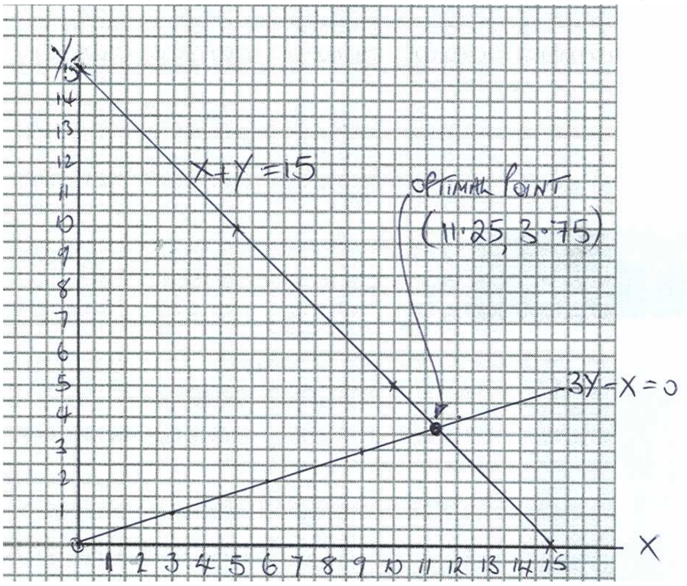
Therefore, P = 2000X + 3000Y

(c) There are two constraints: they cannot produce more than 15 cars in a year and they cannot produce more than 3 lots of Y for every 1 of X.

Constraint 1: X+Y≤15

Constraint 2: 3Y≤X or 3Y-X≤0

The solution is shown below but in summary we find:



X= 11.25, Y = 3.75, P = £33,750

Students might discuss how meaningful the figure 11.25 is: can 0.25 of car be made and sold?

ACTIVITY TWO

Learning Objectives

LO1. Students to consolidate knowledge and understanding of linear programming using a graphical method.

LO2. Students to be able to independently interpret the results from a graphical linear program and to identify limitations.

Task One

Which statements are TRUE and which are FALSE?

Statement or Assertion	Tick (✓)	
	True	False
A linear programme cannot have more than five constraints.		
Linear programming must always result in an economic variable being maximised.		
The feasible area shows all of the possible combinations of attainable solutions.		
It is possible to have more than one optimal solution.		
An objective function is a qualitative expression summarising a goal or target of an economic agent.		
A constraint is a mathematical representation of a limitation or restriction that an economic agent faces.		

Statement or Assertion	Tick (✓)	
	True	False
A firm is usually assumed to be a profit-maximiser but could have other objectives.		
All economic problems can be represented by linear programmes and solved.		

Task Two

Consider the LP problem below.

The CVH Company of Leeds is a revenue maximising firm. It has chosen to focus on the production and sale of two goods: A and B.

Good A requires 1 hour of skilled labour and 4 kilos of metal. It is a standard product and retails for £40.

Good B is a higher quality version of good A. It needs 2 hours of skilled labour and demands 3 kilos of metal. It sells for a slightly higher price of £50 per unit.

Under current operating conditions, the firm is prepared to hire 40 hours worth of labour each day and it has a long term contract with a supplier who can deliver 120 kilos of metal each day.

Your task is to construct a simple LP and graph to solve.

Task Three (Extension Task)

Look back at the LP you constructed for Task Two. One of the key variables was labour. The case study suggested that a maximum of 40 hours of labour was available each day. Suppose this was comprised of 5 workers who could each work 8 hours each day.

(i) What implicit assumptions have we made about these workers?

(ii) Do you think these assumptions are realistic?

(iii) In what cases could labour be assumed to be entirely identical or homegenous?

ANSWERS

Task One

Statement or Assertion	Tick (✓)		Reasoning
	True	False	
A linear programme cannot have more than five constraints.		✓	A linear programme can have a large number of constraints.
Linear programming must always result in an economic variable being maximised.		✓	An LP could result in something being minimised eg. costs.
The feasible area shows all of the possible combinations of attainable solutions.	✓		A feasible area shows all of the possible solutions but usually there is a single point within the feasible area which provides an optimal solution.
It is possible to have more than one optimal solution.	✓		It is possible to have more than one optimal solution eg. where an objective function and a constraint coincide for a range of values.
An objective function is a qualitative expression summarising a goal or target of an economic agent.		✓	No – it is a mathematical and quantitative expression. Sometimes it can be a quantitative expression of something which is qualitative in nature eg. a social cost.

Statement or Assertion	Tick (✓)		Reasoning
	True	False	
A constraint is a mathematical representation of a limitation or restriction that an economic agent faces.	✓		A constraint is an expression of a limitation eg. a household's spending is limited by income.
A firm is usually assumed to be a profit-maximiser but could have other objectives.	✓		Some firms may wish to maximise sales eg. if senior managers are paid on a commission basis they could be more likely to seek to expand sales rather than profits.
All economic problems can be represented by linear programmes and solved.		✓	In reality, few economic problems can be summarised as simply and neatly as an LP would suggest. They can however provide useful and insightful approximations to how 'the real world' operates.

Task Two

The objective function is to maximise revenue (TR) which is

Max TR = 40A + 50B subject to:

① A + 2B =< 40 hours per hour of skilled workers' time

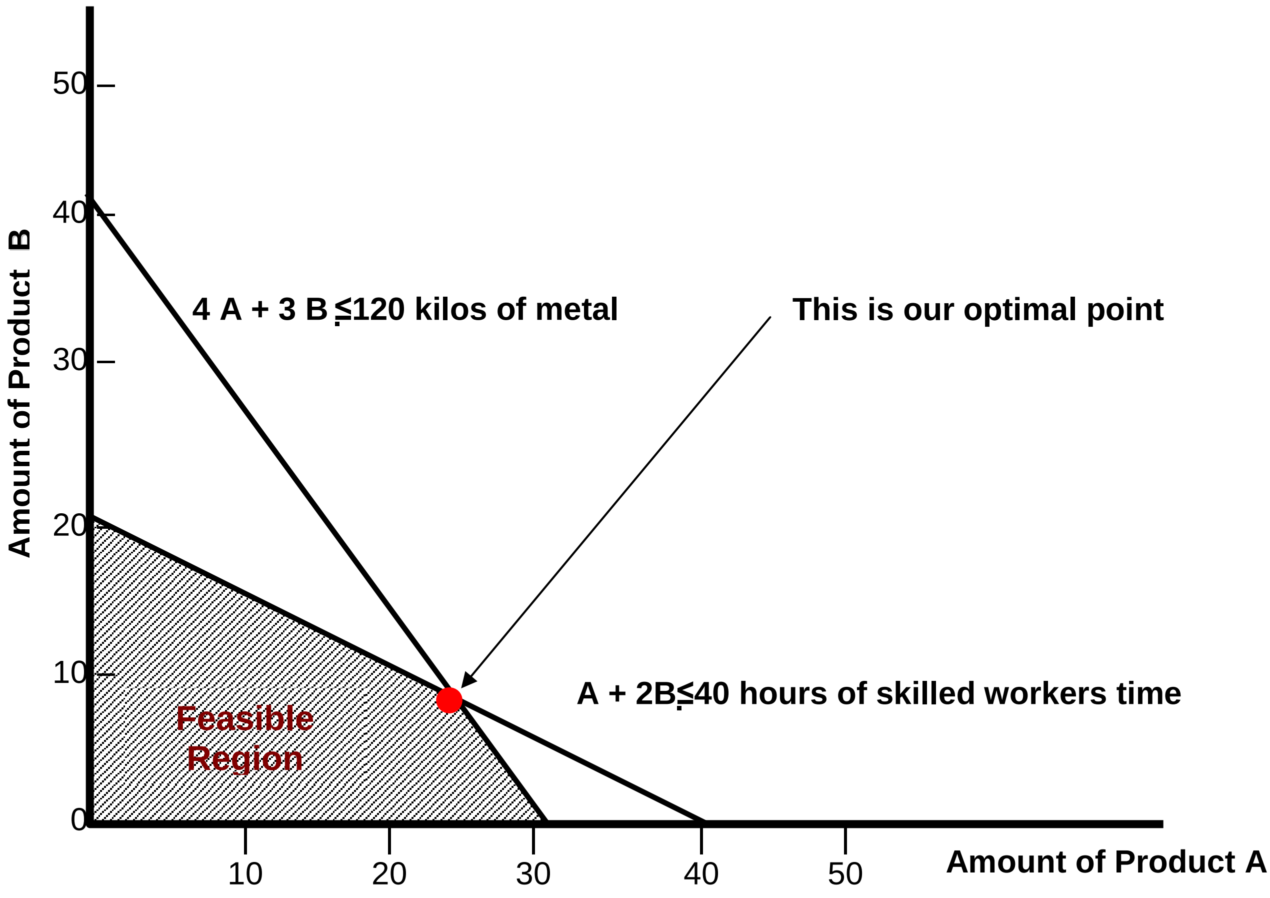
② 4A + 3B =< 120 kilos of metal

③ A and B => 0

The answer is A = 24 and B = 8

Total revenue = (40 * 24) + (50*8) = 960 + 400 = <u>**£1360**</u>

The full graphical solution is shown below:



Task Three

(i) The LP model presupposes that all of the workers are identical: they have the same set of skills they work equally hard and they produce work at the same rate and of the same identical quality.

(ii) The assumption that all of the workers are identical is clearly a strong assumption. Can we ever really say that two workers are equal in every respect. It is rare that two workers could have their productivity precisely measured and even if this could be done it would be very surprising if the results turned out to be exactly equal. Perhaps the real question is not whether the assumption is realistic but whether the assumption really affects the analysis or our conclusions. There, although the assumption is strong it does not have a material impact on our conclusion or our results.

(iii) Labour is rarely, if ever, completely homogenous. It is more likely to be uniform where simple or low level skills are required or where productivity is measured more in terms of the quantity produced by a worker rather than the quality of what they produce. An example could include manual work e.g. digging a hole in the road. Other examples could include ‘mechanical labour’ e.g. the use of robotics in car factory. In this instance, it could be relatively straightforward to measure the quantity and quality of what is produced but again this is likely to be confined to routine or repetitive manual tasks e.g. multiple spot welding in a plant assembly large commercial aircraft.

6. Top Tips

1. Students are likely to benefit from opportunities to translate descriptive objective functions and constraints into mathematical expressions. Lecturers will find that the time they invest in contextualising constraints and objective functions will be repaid: students will grasp the ‘real world’ applications and significance of LP but will also be in a stronger position to attempt any examination questions which be might be required as part of their final assessment.

2. Students can often try and work out what the objective function and constraints are before they have fully read a LP question or problem. This can present real problems with more complex LP problems where significant information is contained later on in a given problem. For example, information might be given about a firm’s profit function but in the final paragraph students are told that the firm is actually a sales maximiser. A student that jumps in early to formulate a profit maximising objective function will have made a small but highly significant error.

3. There is often a temptation to move quickly away from graphical representations of LP to purely mathematical formulations. This transition needs to be managed carefully: any ‘non mathematical’ students can grasp the graphical approach but struggle with the more formal mathematical structure. One way to ease this transition is to map each stage from the graphical to the mathematical i.e. starting with how a descriptive objective function can be graphed and then in turn into translated into a mathematical function and so on.

4. An effective way for students to become expert learners in LP is for them to independently

research and explore ways in which LP can be used in the ‘real world’ and for them to attempt to construct their own ‘personal LPs’. Even if students do not progress beyond a simply descriptive expression of an LP problem, they will have grasped the fundamentals of what a LP problem is and how they could attempt to solve it.

Typical ‘Personal LPs’ could include:

- maximising my score in an exam subject to constraints on my time for revision and sleeping;
- maximising my utility from study and socialising subject to constraints on income and needing to pass exams with a good grade;
- trying to maximise my income subjects to ‘constraints’ that I also need to spend time with my friends and partner!

7. Links with the online question bank

The online question bank offers a wide range of opportunities for students to apply their knowledge of linear programming and also to check their understanding. The online questions can be split into 2 main themes: identifying and matching constraints; and locating optimal points.

Theme 1: Matching inequalities

These questions could be introduced shortly after students have discussed the meaning of constraints and they feel confident translating written constraints into simple inequalities. e.g. to incorporate these online questions after students have undertaken the activity with laminated graph paper and cardboard constraints (see above).This jump from descriptive expression to mathematical formulation is significant for many students. These questions ask students to consider a description of one or more constraints and then to identify which inequalities fit.

It might be helpful for lecturers to project some of the initial questions and to work through an answer, perhaps through use of an interactive whiteboard. Smaller groups might benefit from working collaboratively in the first instance and then perhaps moving to paired or individual work.

These questions could feed-in to a review of understanding or plenary session. Students could be invited to share what they have learnt and, more importantly, to describe how they solved the questions. This could draw upon some effective Assessment for Learning techniques where students evaluate the answers and explanations offered by their peer groups.

Theme 2: Finding an optimum point and ranges of optimal points

The online question bank offers simple questions requiring students to identify optimum points. These questions could be introduced after the graphical solution work has been completed and students are confident focusing on purely mathematical problem sets. Students could initially work in pairs.

These questions can include a case where there are a range of optimal points rather than a single unique optima. This links with one of the ‘true/false’ questions in Activity One. Again, there is considerable scope for students feeding-back **how** they solved the problem rather than simply articulating the final solution.

8. Conclusion

This Guide has attempted to offer some practical strategies to help colleagues deliver Linear Programming , taking account of the wide range of mathematical ability, confidence and attainment which students will probably present.

A key to successful delivery is to build the LP model in simple and discrete stages. First, through the successive explanation of concepts: the objective function, the constraints, the feasible region, the optimal point. Second, there are stages in ‘solution process’: start with the purely descriptive, then to the graphical an then to the entirely mathematical.

Underpinning all of these stages and activities should be a clear and relevant practical application of LP. Without this, there is a real danger that that LP will be viewed merely as a mathematical ritual which students have to endure rather than an illuminating and engaging way to formulate and solve economic problems.