Teaching and Learning

Guide 10:

Matrices

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Section 1: Introduction to the guide

This guide is designed to set out some of the basic mathematical concepts needed to teach economics and financial economics at undergraduate level. The concepts covered by this guide are (i) the dimensions of a matrix and surrounding vocabulary; (ii) addition, subtraction, multiplication and division of matrices; (iii) matrix transposition; (iv) matrix inversion; (v) finding the determinant of a matrix; (vi) Cramer's rule; (vii) Input-Output analysis.

It is very useful to use Excel to assist teaching the topic of matrices. Excel has a large number of in built functions to help find the transpose and inverse of matrices. It also has an inbuilt function to multiply matrices. One key issue in matrix multiplication is "conformability". Excel focuses on "conformability" directly as before you undertake any matrix operations in Excel you need to determine the dimension of the resultant matrix and highlight a selection of cells matching this dimension. If you highlight an incorrect dimension Excel is unable to undertake the calculation.

The use of Excel is an essential tool for anyone working in finance. Throughout this guide Excel screenshots and links to files are provided. It would be useful therefore if the session utilising this material were presented in a classroom where students can gain hands on experience.

Matrices are commonly used in finance. As a consequence a number of the examples have a finance bias. These include (i) using matrices to calculate a covariance matrix; (ii) using matrices to calculate the risk of a share portfolio. An example of how matrices are used in a journal article is included as a teaching and learning activity. This is an excellent way of demonstrating to students that learning mathematical techniques is not simply a case of learning for the sake of learning. It is not always possible to find appropriate examples in journal articles but the one included in this guide is set at a suitable level. The lecturer is also directed to an alternative article for an exercise that could be used as a tutorial or examination question.

With the use of Excel for matrix multiplication and inversion it is less apparent on the relative advantage of using Crammers rule over standard techniques to find solutions to problems. An algebraic based example is included to show that Crammers rule is still useful. This topic is

most definitely a "doing" topic. Consequently a large number of examples are included to help the lecturer.

Section 2: Definitions and Operations

1. The concept of definitions and operations

Matrices are a difficult topic for many students and a set of clear definitions are very important. These will need to be revisited to ensure students have a secure understanding of the key terms. Some definitions that might be useful include:

a) Defining a matrix

A <u>matrix</u> is a rectangular array of numbers, parameters or variables arranged in some meaningful order. The elements (or parameters or variables) are referred to as the <u>elements</u> of a matrix. The elements in a horizontal line constitute a <u>row</u> of the matrix and it follows that the elements in a vertical line constitute a <u>column</u> of the matrix. The entries in a matrix are usually enclosed in two curved lines or square brackets. Thus the general matrix with m rows and n columns can therefore be written as:

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & ... & a_{1n} \\ a_{21} & a_{22} & ... & ... & a_{2n} \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & .... & a_{mn} \end{pmatrix}$$

The element in the i'th row and the j'th column is a_{ij}. If we call the matrix above, A, we can sometimes avoid writing the matrix out in full, and instead write, very succinctly, A.

b) Defining the dimensions

A matrix, like the one above, with m rows and n columns is called an "m by n" or an "m x n" matrix. This determines the <u>dimensions</u> of the matrix.

Lecturers could remind students that in our example that m is the number of rows and n is the number of columns and also to be clear that the row number always precedes the column number.

2. Presenting the concept of definitions and operations

It is easy to present matrices as a purely abstract and theoretical concept. The danger of such an approach is that the mathematics can be perceived by students to become and end in itself; that the point of learning about matrices is simply to practise a methodology and pass an exam.

The most effective presentations of matrices will continually contextualise matrices and give real world examples to support the conceptual framework. In some ways, with a 'hard' topic like matrices the lecturer might even find that there is a real imperative to put matrices into an applied setting almost as a way to convince students that it is valid, that matrices are a legitimate tool which can inform economic problems.

For example, the following example is intended to show that matrices are a compact way of articulating mathematical problems.

Consider the risk (measured by variance) of a two asset stock portfolio can be written as:

$$\sigma_p^2 = x_1^2 \sigma_1^2 + x_2^2 \sigma_2^2 + 2x_1 x_2 \sigma_{12}$$

where :

x_1 is the proportion of wealth invested in asset 1,

x_2 is the proportion of wealth invested in asset 2,

σ_1^2 is the variance of asset 1,

σ_2^2 is the variance of asset 2,

σ_{12} is the covariance between assets 1 and 2.

This formula can be extended to 2, 3 or n assets. However as we extend the number of assets the number of terms gets larger. In particular if we have n assets there a n(n-1)/2 covariance terms to write down, i.e. n=100 there are 100x99/2 = 4950 covariance terms. Therefore expressing this equation in its linear form can become problematic as the number of assets gets large.

However this equation can be generalised to m assets and written more compactly using

$$\sigma_p^2 = x' \sum x$$

matrices. where :

      x is a vector of portfolio weights (wealth invested),

      $\sum$ is a matrix of covariances between assets

Hopefully, students will agree that the second representation is easier to remember!

3. Delivering the concept of definitions and matrix operations and to small or larger groups

In terms of delivering the 'nuts and bolts' of basic matrix operation, there is little substitute for actually going through the methods with students either in a lecture or tutorial setting. Students will need to see clear examples of addition, subtraction, etc., and a brief overview is provided below to help colleagues deliver this introductory material. This is not intended to replace a good textbook approach but could inform a set of PowerPoint slides or lecture handout.

Matrix operations

Addition:

Remind students that with regular algebra we could write c = a + b, whereas with matrix algebra we would write C = A + B. They will need to know that in order to do this, matrices A and B must be of the same dimension. If this is the case then each element of one matrix is added to the corresponding element of the other matrix. Thus if A and B are both 2 x 2 matrices element a_{11} will be added to b_{11}, a_{21} added to b_{21}, a_{12} added to b_{12} and a_{22} added to b_{22}.

Addition: Worked Example

$$A = \begin{pmatrix} 7 & 4 \\ 6 & 4 \end{pmatrix}, B = \begin{pmatrix} 3 & 9 \\ 8 & 2 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 7+3 & 4+9 \\ 6+8 & 4+2 \end{pmatrix} = \begin{pmatrix} 10 & 13 \\ 14 & 6 \end{pmatrix}$$

Subtraction:

Again, students could look at "regular algebra" where we could write c = a - b, whereas with matrix algebra we would write C = A - B. However in order to do this, matrices A and B must be of the same dimension. If this is the case then each element of one matrix is subtracted from the corresponding element of the other matrix. Thus if A and B are both 2 x 2 matrices element b_{11} will be subtracted from to a_{11}, b_{21} subtracted from a_{21}, b_{12} subtracted from a_{12} and b_{22} subtracted from a_{22}.

Subtraction: Worked Example

$$A = \begin{pmatrix} 7 & 4 \\ 6 & 4 \end{pmatrix}, B = \begin{pmatrix} 3 & 9 \\ 8 & 2 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 7-3 & 4-9 \\ 6-8 & 4-2 \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ -2 & 2 \end{pmatrix}$$

Scalar Multiplication

Students will need to be clear that in linear algebra a real number such as 9, -3 or 0.06 is called a <u>scalar</u>. Moreover, they will need to see that the multiplication of a matrix by a scalar involves multiplication of each element of the matrix by the scalar and that this type of multiplication is easy to remember as it scales the matrix up or down according to the value of the matrix.

Scalar Multiplication: Worked Example

$$A = \begin{pmatrix} 7 & 4 \\ 6 & 4 \end{pmatrix}, k = 4$$

$$Ak = \begin{pmatrix} 28 & 16 \\ 24 & 16 \end{pmatrix}$$

The Excel file containing the solutions to these three problems can be found online at www.lawseconomics.co.uk/metal/examples1to3.xls

Lecturers might want to flag up to students that in order to undertake matrix addition, subtraction and scalar multiplication in Excel you must first of all define the name of the matrix or vector. For example, in this spreadsheet we must first of all highlight cells B1:C2 then select "Insert", "Name", "Define" and call this range of cells A. Repeat for B.

Then when it comes to adding A to B, the result of which will also be a 2 x 2 matrix, you must first of all highlight a 2 x 2 block of cells and then type "=A+B". But rather than pressing "Enter" you must press "CTRL+SHIFT+ENTER" together to obtain the answer.
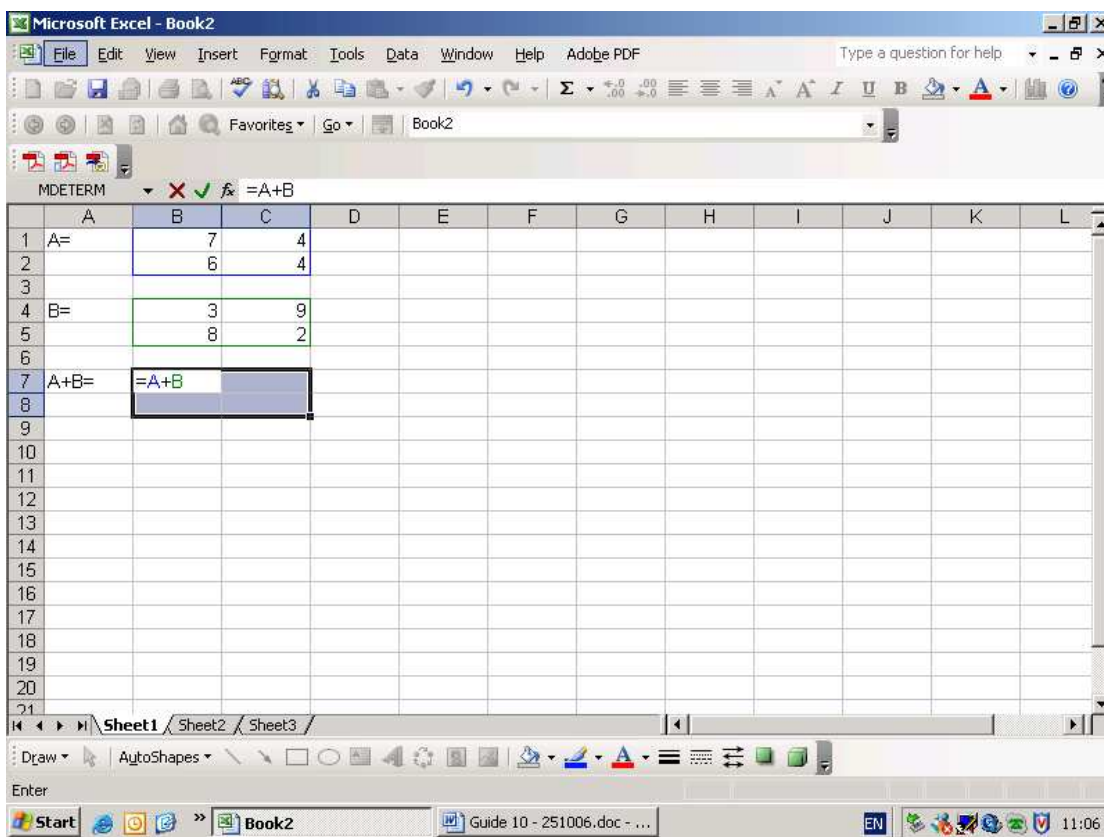
Multiplication

It is often a useful starting point to state that it is often easiest with matrix or vector multiplication to ask yourself what the answer will be first of all. Students will need to know that matrix multiplication in Excel requires this as you have first of all highlight the cells where the answer will be displayed.

Students could be referred back to the original example where the dimensions of a matrix were stated as "m x n" where m is the number of rows and n is the number of columns. The next

step could be to explain that to multiply a (m x n) matrix by another (n x m) then the resultant matrix would be "m x m" and the multiplication operation has only been possible since the number of columns in the first matrix (the "lead" matrix) is equal to the number of columns in the second matrix (the "lag" matrix).

Hint: A suggestion as a way to remember this made by Dowling (1993) is to place the two sets of dimensions in the order in which they are to be multiplied. Then circle (mentally or physically) the last number of the lead matrix dimensions and the first number of the lag matrix dimensions. If the two numbers are equal then they are "conformable" (i.e. matrix multiplication is possible in the given order). Furthermore, the outer numbers will provide the dimensions of the product matrix.

Multiplication: Worked example

Below we consider the multiplication of matrix A by matrix B, i.e. A x B.

Dimensions of matrix A	Dimensions of matrix B	Is A x B possible?	Dimensions of resultant matrix
(2 x 2)	(2 x 2)	Yes, number of columns in A (2) equals the number of rows in B (2)	(2 x 2) - 2 rows and 2 columns
(2 x 2)	(2 x 3)	Yes, number of columns in A (2) equals the number of rows in B (2)	(2 x 3) - 2 rows and 3 columns
(1 x 4)	(4 x 1)	Yes, number of columns in A (4) equals the number of rows in B (4)	(1 x 1) - a scalar
(5 x 1)	(1 x 5)	Yes, number of columns in A (1) equals the number of rows in B (1)	(5 x 5) - 5 rows and 5 columns
(4 x 2)	(2 x 1)	Yes, number of columns in A (2) equals the number of rows in B (2)	(4 x 1) - 4 rows and 1 column
(2 x 2)	(3 x 2)	No, number of columns in A (2) is not equal to the number of rows in B (3)	N/A

(1 x 4)	(1 x 4)	No, number of columns in A (4) is not equal to the number of rows in B (1)	N/A
(5 x 1)	(5 x 1)	No, number of columns in A (1) is not equal to the number of rows in B (5)	N/A

Links with the online question bank

General questions on matrices can be found at:

http://www.metalproject.co.uk/METAL/Resources/Question_bank/Matrices/index.html and questions focused specifically on multiplication are also located at:

http://www.metalproject.co.uk/METAL/Resources/Question_bank/Matrices/index.html

4. Discussion Questions

Students could be asked to research some business or economics data which they present as a simple matrix e.g. share data, GDP figures over time etc

5. Activities

ACTIVITY ONE

Learning Objectives

LO1: Students learn how to complete matrix multiplication

LO2: Students learn how to use Excel to complete matrix multiplication

LO3: Students learn how to use the matrix operators

Task One

If: $A = \begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$

Calculate (a) AB and (b) BA.

Task Two

An excellent way of reinforcing students understanding of matrix multiplication is to use Excel. The appropriate function is "MMULT". There are two key differences when using this function compared to the majority of Excel functions:

Rather than pressing "Enter" or clicking "OK" if using the function wizard you must press "CTRL"+"Shift"+"Enter" together to tell Excel to undertake the calculations.

Since the outcome of this function is usually going to be something other than a scalar you must inform Excel where to display the output. Thus if the outcome is a (2 x 1) matrix you must first of all highlight 2 rows and 1 column before starting to use the function.

Students might wish to see a video showing these operations being undertaken using Excel www.lawseconomics.co.uk/metal/mmult.zip

Task Three

If: $A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

Calculate BA.

ANSWERS

ACTIVITY ONE

Task One

Remembering what we have learned previously and following the helpful hint:

$$(2 \times 2) \times (2 \times 2)$$

$$(2 \times 2)$$

Therefore whichever way we perform the multiplication the result will be a 2 x 2 matrix.

$$AB = \begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3x3 + 6x2 & 3x2 + 6x4 \\ 1x3 + 4x2 & 1x2 + 4x4 \end{pmatrix} = \begin{pmatrix} 21 & 30 \\ 11 & 18 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 3x3 + 2x1 & 3x6 + 2x4 \\ 2x3 + 4x1 & 2x6 + 4x4 \end{pmatrix} = \begin{pmatrix} 11 & 26 \\ 10 & 28 \end{pmatrix}$$

Task Two

-

Task Three

$$BA = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 & 4 \\ 3 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2x2 + 3x3 & 2x1 + 3x2 & 2x4 + 3x0 \\ 1x2 + 4x3 & 1x1 + 4x2 & 1x4 + 4x0 \end{pmatrix} = \begin{pmatrix} 13 & 8 & 8 \\ 14 & 9 & 4 \end{pmatrix}$$

6. Top Tips

Tip 1: Special cases

Lecturers will want to ensure that the 'special cases' are fully understood. That is:

A square matrix – in a square matrix the number of rows equals the number of columns. A square matrix differs from other matrices in that it has diagonals. The diagonal leading from top left to bottom right is called the main diagonal. A square matrix which has all its elements zero except for those along its main diagonal, and these are all ones, is called a unit matrix or identity matrix.

A matrix composed of several rows and a single column has dimensions m x 1 and is referred to as a **column vector.**

A matrix composed of a single row and several columns has dimensions 1 x n and is referred to as a **row vector.**

This could be presented in a simple template or perhaps as a spider diagram.

Tip 2: Multiplication

When approaching this topic is far easier to let students become acquainted with multiplying vectors before they move onto multiplying matrices. Before attempting matrix multiplication students should consider will the dimensions of the resultant matrix be?

7. Conclusion

This preparatory work will be crucial to the subsequent success students achieve with higher level and more complex matrix work. Any scheme of work should ensure that plenty of time is given for students to practise these basic operations and for one-to-one support to be given to remedy any initial problems.

Section 3: Transposing and Inverting a Matrix and Matrix Determinants

1. The concept of transposition, inversion and matrix determinants

As noted earlier, this is a topic which students will perceive as difficult. This is compounded by the rather opaque terminology used and one key responsibility incumbent upon lecturers is to help students make sense of these labels and concepts.

a) Transposition

For example to show that the term **transposition** is actually something quite simple as:

If the rows and columns of a matrix are interchanged the new matrix is known as the transpose of the original matrix. E.g.

If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then the transpose, $A' = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

This could even be introduced using images rather than numbers such as:

If Z=

Then Z' =

And then more formally that

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ then $A' = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$

b) Determinants

Similarly, lecturers could use a simple but formal overview by stating that the determinant is a square group of numbers set out in matrix format but enclosed by straight rather than curved

lines. It is important that students have mastered find the determinant of a 2 x 2 matrix before progressing to 3 x 3 and above.

The determinants of orders 1 and 2 are defined as:

$$\left|a_{11}\right| = a_{11}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

For example, finding the value of the following determinants of order 2:

$$\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 3x5 - (-2)x4 = 15 + 8 = 23$$

$$\begin{vmatrix} -1 & 6 \\ 0 & 4 \end{vmatrix} = -1x4 - 6x0 = -4$$

$$\begin{vmatrix} a-b & b \\ b & a+b \end{vmatrix} = (a-b)(a+b) - b^2 = a^2 - b^2 + ab - ab - b^2 = a^2 - 2b^2$$

c) Inversion

Similarly, suppose we have a 2 x 2 matrix A:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then the inverse of A is:

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Hence to obtain the inverse of a square matrix of order 2 we:

Step 1: interchange the elements on the main diagonal,

Step 2: change the sign of the other two elements,

Step 3: divide by the determinant corresponding to the original matrix.

This is a simple teaching sequence of core concept but one through which students will need to be guided carefully.

2. Presenting the concept of transposition, inversion and matrix determinants

Transpositions are fairly straightforward to teach and lecturers might want to focus on inversions and matrix determinants. An outline of material that could be presented is set out below

Inversions

Inversions are something which students need to practise. Students can be reminded that in traditional arithmetic division is the reciprocal or "inversion" of multiplication.

If we consider say 1 divided by 4, to take a simple example, we say that 4 x ¼ = 1 and ¼ x 4 = 1. In matrix algebra we define inversion using the same approach. We say that the inverse of a square matrix A is another square matrix B where:

$$AB = I \text{ and } BA = I$$

where I is the identity matrix of the same order.

Such a matrix B, if it exists, is unique and is called the inverse of A (written as A^{-1}).
Consequently we have:

$$AA^{-1} = I \text{ and } A^{-1}A = I$$

Students could consider these two square matrices:

$$\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix} \text{ and } \begin{pmatrix} 7 & -10 \\ -2 & 3 \end{pmatrix}$$

Multiplying these two together we get:

$$\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} 7 & -10 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 3x7 + 10x(-2) & 3x(-10) + 10x3 \\ 2x7 + 7x(-2) & 2x(-10) + 7x3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Or,

$$\begin{pmatrix} 7 & -10 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} 7x3 + (-10)x2 & 7x10 + (-10)x7 \\ (-2)x3 + 3x2 & (-2)x10 + 3x7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Clearly these two matrices are inverses of one another and students need to understand this technique

Determinants

Students will need to be aware of the process for calculating determinants including the notion of co-factors. Again, some useful material is provided below which can be incorporated into notes or slides.

A discussion of the determinants of a 3 x 3 matrix will need to first introduce the concept of cofactors. Students could be shown that Corresponding to each element a_{ij}, of a matrix, A, there is a cofactor, A_{ij}. And that it follows that a 3x3 matrix has nine elements, so there are nine cofactors, to be computed.

The cofactor, A_{ij}, is defined to be the <u>determinant</u> of the 2x2 matrix obtained by deleting row i and column j of A, prefixed by either a "+" sign or as "-" sign.

The prefixes are assigned according to the following pattern:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

For example, suppose we wish to calculate A_{32}, which is the cofactor associated with a_{32} in the matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The element a_{32} lies in the third row and second column. Consequently we delete the third row and the second column to produce the 2 x 2 matrix:

$$\begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix}$$

The cofactor A_{32}, is the determinant of this 2 x 2 matrix prefixed by a "-" sign due to the assigning pattern above, i.e.

$$A_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = -\left(a_{11}a_{23} - a_{13}a_{21}\right) = -a_{11}a_{23} + a_{13}a_{21}$$

Now we have introduced the cofactor concept we can move onto showing the determinant of a 3x3 matrix. The determinant of the general 3 x 3 matrix is as follows:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32}$$

Students should note that every one of these six products of elements contains exactly one element from each row and exactly one element from each column.

3. Delivering the concept of transposition, inversion and matrix determinants to small or larger groups

Lecturers could deliver this material through a series of formal lectures but supplement with student-led tutorials. For example, students could be invited to sign-up to deliver 'mini-lectures'

covering one or more these topics and other students are required to sign-up to attend at least two as 'audience members'. This can be a simple but effective technique to help students grasp the essentials of matrix algebra; if a student can deliver a concise and cogent overview then they are very likely to have understood the material. This teaching strategy could be differentiated by pairing higher ability with lower ability students to lead each workshop.

Links with the online question bank

Questions on determinants can be found on the METAL website at:

http://www.metalproject.co.uk/METAL/Resources/Question_bank/Matrices/index.html

whilst tasks centred on matrix inversion are available from:

http://www.metalproject.co.uk/METAL/Resources/Question_bank/Matrices/index.html

4. Discussion Questions

Students could be encouraged to bring any problem sets they have been working on into tutorial and to discuss any problems they have encountered. This could identify to the lecturer or tutor any common misunderstandings or areas of difficulty and inform the tutor's tutorial scheme of work. This 'co-construction' of future teaching and learning objectives is an important part of a successful Assessment for Learning strategy.

5. Activities

ACTIVITY ONE

Learning Objectives

LO1: Students have opportunity to refresh calculation of averages

LO2: Students to learn how to calculate averages using matrices

LO3: Students learn how matrices can be used in applied economics

Consider the data below which are the weekly stock price returns of three UK gambling stocks:

	SPORTINGBET	888 HOLDINGS	PARTYGAMING
17/07/2006	-18.79	-6.65	-2.83
24/07/2006	-29.43	-18.13	1.46
31/07/2006	23.49	-1.90	4.78

07/08/2006	17.60	7.42	9.13
14/08/2006	-19.12	-9.31	-11.92
21/08/2006	4.81	-2.65	8.79
28/08/2006	-3.78	2.72	-0.87
04/09/2006	7.53	4.14	5.07
11/09/2006	-43.20	-9.70	-10.48
18/09/2006	19.10	4.93	0.00
25/09/2006	2.62	-6.71	-5.85
02/10/2006	-62.50	-22.12	-55.22

The average weekly return is:

Sporting Bet = -8.47%

888 Holdings = -4.83%

Partygaming = - 4.83%

Task One

Using matrices how would I find the average return of Sporting Bet?

Task Two

Confirm that the average of both 888 Holdings and Partygaming is -4.83%

ACTIVITY TWO

Learning Objectives

LO1: Students learn the meaning and significance of covariance

LO2: Students learn how to calculate variance using matrix algebra

LO3: Students learn how to apply matrix algebra to solve an applied economics problem

LO4: Students learn how to make an economic inference using covariance information

Background information

The variance of a portfolio can be calculated as:

$$\sigma_p^2 = x' \sum x$$

where :

x is a vector of portfolio weights (wealth invested),

$\sum$ is a matrix of covariances between assets

For a four asset case this would look like the following when expanded out:

$$\sigma_p^2 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

where :

x_i is the weight invested in asset i,

σ_{ij} is the covariance between asset i and j,

Note here we first of all multiply a (1x4) row vector by a (4x4) matrix to obtain a (1x4) row vector which is multiplied by a (4x1) column vector to give a (1x1) answer.

Task One

Let us assume the following covariances between four UK stocks:

Covariance	MMO^2	Marks and Spencer	Barclays	Scottish Power
MMO^2	27.30	2.10	3.60	-0.08
Marks and Spencer	2.10	15.92	3.54	1.62
Barclays	3.60	3.54	13.33	1.88
Scottish Power	-0.08	1.62	1.88	7.02

Calculate the variance of an equally weighted (x_i=0.25) portfolio using matrix algebra

Task Two

What can you infer about risk?

ACTIVITY THREE

Learning Objectives

LO1: Students learn how to find the inverse of a matrix

LO2: Students learn how to calculate the determinant

Task One

Find the inverse of the following matrix

$$A = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}$$

Task Two

Find the determinant of the following 3 x 3 matrix:

$$|A| = \begin{vmatrix} 5 & 4 & 9 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{vmatrix}$$

Task Three

Find the inverse of the following matrix:

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$$

ACTIVITY FOUR

Learning Objectives

LO1: Students learn how to independently calculate determinants

LO2: Students learn how to make inferences about whether a matrix is singular or not

LO3:

Task One

Find the determinant of the following 3x3 matrix:

$$|A| = \begin{vmatrix} 20 & 5 & 5 \\ 15 & 10 & 5 \\ 5 & 0 & 1 \end{vmatrix}$$

Task Two

Find the determinant of the following 4x4 matrix:

$$|A| = \begin{vmatrix} 2 & 2 & 4 & 9 \\ 4 & 1 & 0 & 2 \\ 4 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 \end{vmatrix}$$

ANSWERS

ACTIVITY ONE

Task One

An average is found by adding up a series of numbers and dividing by the number that they are. So for the series above there are twelve numbers so we add them together and divide by 12. To replicate the figure of -8.47% above using matrices we would multiply:

$$\left(\tfrac{1}{12} \ \tfrac{1}{12} \ \tfrac{1}{12} \ \tfrac{1}{12} \ \tfrac{1}{12} \ \tfrac{1}{12} \ \tfrac{1}{12} \ \tfrac{1}{12} \ \tfrac{1}{12} \ \tfrac{1}{12} \ \tfrac{1}{12} \ \tfrac{1}{12} \right) \begin{pmatrix} -18.79 \\ -29.43 \\ 23.49 \\ 17.60 \\ -19.12 \\ 4.81 \\ -3.78 \\ 7.53 \\ -43.20 \\ 19.10 \\ 2.62 \\ -62.50 \end{pmatrix} = -8.47$$

Task Two

Students should confirm through matrix algebra that the assertion is indeed correct.

ACTIVITY TWO

Task One

$$\sigma_p^2 = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix} \begin{pmatrix} 27.30 & 2.10 & 3.60 & -0.08 \\ 2.10 & 15.92 & 3.54 & 1.62 \\ 3.60 & 3.54 & 13..33 & 1.88 \\ -0.08 & 1.62 & 1.88 & 7.02 \end{pmatrix} \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{pmatrix}$$

Note that the diagonal elements here are the individual variances of the four stocks. We can see that MMO^2 is the most volatile stock. Interestingly, we also see that MMO^2 and Scottish Power have negative covariance.

Using the "MMULT" command in Excel we can find the answer very easily:

x'=	0.25	0.25	0.25	0.25
□=	27.30	2.10	3.60	-0.08
	2.10	15.92	3.54	1.62
	3.60	3.54	13.33	1.88
	-0.08	1.62	1.88	7.02
x=	0.25			
	0.25			
	0.25			
	0.25			
x'□=	8.23	5.80	5.59	2.61
x'□x=	5.56			

Task Two

Notice that the answer of 5.56 is less than any of the individual variances and hence we can reduce risk by putting 25% of our wealth in each asset rather than "putting all our eggs in one basket".

This spreadsheet is available at http://www.lawseconomics.co.uk/metal/covariance.xls

Note when carrying out these calculations using Excel it is possible to "nest" functions as follows:
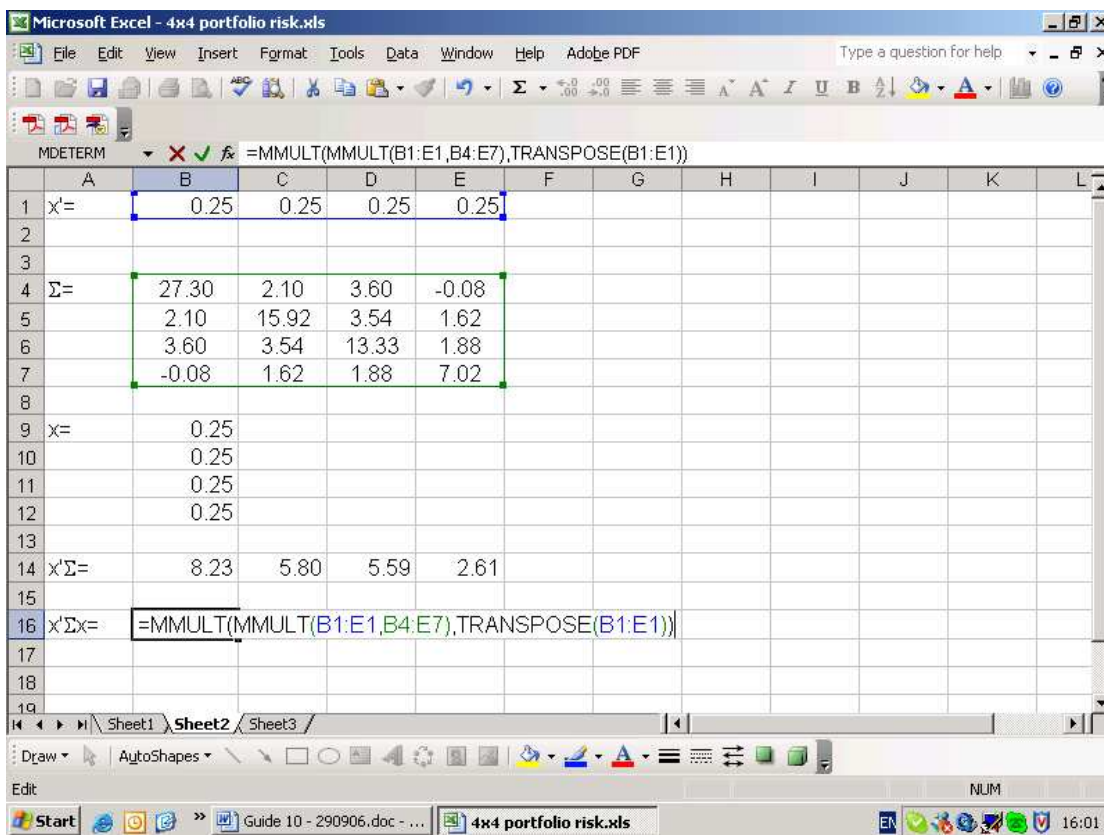
Note: Rather than referring to an array of cells as say B4:E4 (as in the screen shot above) it is possible to name the range of cells (Using "Insert", "Name", "Define"). You could therefore replace B4:E4 with "sigma" in the example above.

ACTIVITY THREE

Task One

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} 3 & -5 \\ -2 & 3 \end{pmatrix} = \frac{1}{3x3 - 5x2}\begin{pmatrix} 3 & -5 \\ -2 & 3 \end{pmatrix} = -1\times\begin{pmatrix} 3 & -5 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 5 \\ 2 & -3 \end{pmatrix}$$

Task Two

Expanding by the 3rd row (due to the presence of a zero):

$$|A| = -3\begin{vmatrix} 5 & 9 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = -3(5x1 - 9x1) + (5x2 - 4x1) = 12 + 6 = 18$$

Task Three

Task Three

First of all finding the determinant by expanding by column 1:

det(A) = 1 x (16-9) – 1 x (12 – 9) +1 x (9-12) = 7 -3 -3 = 1

Then finding the nine cofactors:

$$A_{11} = +(16 - 9) = 7$$
$$A_{21} = -(12 - 9) = -3$$
$$A_{31} = +(9 - 12) = -3$$
$$A_{12} = -(4 - 3) = -1$$
$$A_{22} = +(4 - 3) = 1$$
$$A_{32} = -(3 - 3) = 0$$
$$A_{13} = +(3 - 4) = -1$$
$$A_{23} = -(3 - 3) = 0$$
$$A_{33} = +(4 - 3) = 1$$

Putting these in-order forms the <u>adjugate</u> matrix:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

Then taking the transpose of the adjugate matrix to form the <u>adjoint</u> matrix:

$$\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

This was a very straight forward task due to the symmetry of the adjugate matrix.

The inverse is therefore:

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = 1\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Why have we done all this? Hopefully, students will be questioning why there has been so much emphasis on finding the determinants and inverses of matrices. In order to reinforce that it is time well spent we should use a number of examples.

ACTIVITY FOUR

Task One

Expanding by the 3rd row (due to the presence of a zero):

$$|A| = +5\begin{vmatrix} 5 & 5 \\ 10 & 5 \end{vmatrix} + 1\begin{vmatrix} 20 & 5 \\ 15 & 10 \end{vmatrix} = +5(5x5 - 5x10) + (20x10 - 5x15) = -125 + 125 = 0$$

The value of the determinant is zero. This implies that the corresponding matrix is <u>singular</u> and hence has no inverse.

Task Two

The first thing to ask if the pattern of "+'s" and "-'s". This is just an extension of the pattern introduced earlier:

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

The next thing to consider is what row or column is it optimal to "expand" by. If we look closely we will see that row 3 and column 3 both have two zero's which will reduce the number of calculation. Therefore we should choose either of those.

Expanding by column 3:

$$|A| = 4 \times \begin{vmatrix} 4 & 1 & 2 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 2 & 9 \\ 4 & 1 & 2 \\ 4 & 1 & 0 \end{vmatrix}$$

We now have to find the determinants of the 3x3 cofactor matrices.

Consider the term before the minus sign and expanding by row 2:

$$4 \times \left\{ -4 \times \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 1 \times \begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} \right\}$$

$$= 4 \times \left\{ -4 \times (1x1 - 2x2) + 1(4x1 - 2x3) \right\}$$
$$= 4 \times \left\{ -4 \times -3 + 1x(-2) \right\} = 4x(12 - 2) = +40$$

Now considering the term after the minus sign and expanding by row 3:

$$-1 \times \left\{ +4 \begin{vmatrix} 2 & 9 \\ 1 & 2 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 9 \\ 4 & 2 \end{vmatrix} \right\}$$

$$= -1 \times \left\{ +4(2x2 - 9x1) - 1 \times (2x2 - 9x4) \right\}$$

$$= -1 \times \{+ 4(- 5) - 1 \times (- 32)\}$$
$$= -1 \times \{- 20 + 32\} = -12$$

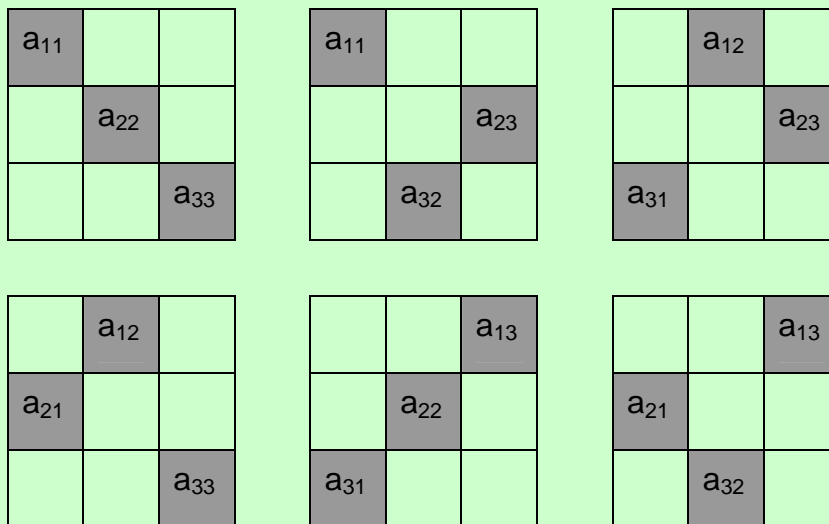The determinant of this 4 x 4 matrix is therefore 40-12 = 28.

6. Top Tips

Tip 1:

Demonstrate to students that It is also possible to find the inverse of matrices using Excel. Again you must highlight the cells where you would like the result to display. The appropriate function is "=MINVERSE".

Tip 2:

Students might find it helpful to think of cofactors using a simple shaded diagram and to then see the pattern becomes evident. In the table below we start in the top left cell (representing element a_{11}) and then highlight a cell in a different row and different column (a_{22}). By default the only remaining cell to choose that is from a different column and a different row is the bottom right (representing element a_{33}). We then start again with the same top left cell highlighted but select a different second row cell (a_{23}) which in turn forces us to select a_{32} as the third row cell. We continue this process until we have exhausted all the potential combinations that have a cell from a different column and row highlighted.

a_{11}		
	a_{22}	
		a_{33}

a_{11}		
		a_{23}
	a_{32}	

	a_{12}	
		a_{23}
a_{31}		

	a_{12}	
a_{21}		
		a_{33}

		a_{13}
	a_{22}	
a_{31}		

		a_{13}
a_{21}		
	a_{32}	

The algebraic expression above can be factored out to become:

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Hopefully we recognised that the terms inside the brackets are the cofactors of the elements outside the brackets. We should also recognise the pattern of 1's and -1's from the table above.

Tip 3:

With many topics in mathematics there exists more than one method to find the solution to a problem. One method popular with students for finding the determinant of a 3x3 matrix is as follows (using the example above):

$$|A| = \begin{vmatrix} 5 & 4 & 9 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{vmatrix}$$

A shortcut involves adding a copy of the first and second rows as follows:

$$\begin{pmatrix} 5 & 4 & 9 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \\ 5 & 4 & 9 \\ 1 & 2 & 1 \end{pmatrix}$$

And then multiplying as follows:

```
5          4          9
1          2          1
0          3          1   5 x 2 x 1 = 10  ⎫
5          4          9   1 x 3 x 9 = 27  ⎬ sum = 37
1          2          1   0 x 4 x 1 = 0   ⎭
```

And:

```
5          4          9   0 x 2 x 9 = 0   ⎫
1          2          1   5 x 3 x 1 = 15  ⎬ sum=19
0          3          1   1 x 4 x 1 = 4   ⎭
5          4          9
1          2          1
```

The difference between these is then 37-19 = 18, which as we found earlier is the determinant of the matrix.

me:tal

You could ask the students why this works. The answer of course is that this is simply an alternative factorisation of the expression given above.

7. Conclusion

Create opportunities in a scheme if work for students to work through problem sets in pairs. This can be a helpful tool to raise any problems or queries that students might have. This topic draws upon and pulls together many different but complementary threads and students are likely to benefit from this collaboration.

Section 4: Cramer's Rule

1. The concept of Cramer's rule

A simple statement at the beginning of a lecture will prove helpful. Perhaps:

The method offered by Cramer's rule allows us to finds the value of one variable at a time and this method requires less effort if only a selection of the variables is required.

Crammers rule for solving any n x n system, Ax=b, states that the i'th variable, x_i, can be found from:

$$x_i = \frac{|A_i|}{|A|}$$

Where A_i is the n x n matrix formed by replacing the i'th column of A by the right-hand side vector, b.

2. Presenting the concept of Cramer's rule

Students will want to see a simple presentation of how Cramer's rule works in practice. One strategy for doing this is set out below:

Consider the simple 2x2 system considered previously:

$$\begin{pmatrix} -7 & 14 \\ 3 & 12 \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} -42 \\ 90 \end{pmatrix}$$

And suppose that we need to find the value of the second variable, Q.

According to Crammers rule:

$$Q = \frac{|A_2|}{|A|}$$

Where
$$A = \begin{pmatrix} -7 & 14 \\ 3 & 12 \end{pmatrix} \text{ and hence } |A| = -7x12 - 3x14 = -84 - 42 = -126$$

$$\text{and } |A_2| = \begin{pmatrix} -7 & -42 \\ 3 & 90 \end{pmatrix} \text{ and hance } |A_2| = -7x90 - 3x(-42) = -630 + 126 = -504$$

Therefore Q = -504/-126 = 4

3. Delivering the concept of Cramer's rule to small or larger groups

A worked example is offered below to help colleagues delivering the concept. Smaller groups will probably have more scope to discuss each step outlined:

Worked example of Cramer's rule

Consider the following three interconnected markets:

Market 1: $Q^S_1 = 6P_1 - 8$ and $Q^D_1 = -5P_1 + P_2 + P_3 + 23$

Market 2: $Q^S_2 = 3P_2 - 11$ and $Q^D_2 = P_1 - 3P_2 + 2P_3 + 15$

Market 3: $Q^S_3 = 3P_3 - 5$ and $Q^D_3 = P_1 + 2P_2 - 4P_3 + 19$

These markets are simultaneously in equilibrium when quantity supplied equals quantity demanded in all three markets:

Market 1: $6P_1 - 8 = -5P_1 + P_2 + P_3 + 23$

Market 2: $3P_2 - 11 = P_1 - 3P_2 + 2P_3 + 15$

Market 3: $3P_3 - 5 = P_1 + 2P_2 - 4P_3 + 19$

Rearranging these expressions so that they can be written in matrix form:

Market 1: $11P_1 - P_2 - P_3 = 31$

Market 2: $-P_1 + 6P_2 - 2P_3 = 26$

Market 3: $- P_1 - 2P_2 + 7P_3 = 24$

Hence, in matrix form:

$$\begin{pmatrix} 11 & -1 & -1 \\ -1 & 6 & -2 \\ -1 & -2 & 7 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} 31 \\ 26 \\ 24 \end{pmatrix}$$

Or A**x**=b.

Using the matrix inversion approach we would be required to invert the 3x3 matrix A, which involves finding its determinant and the nine cofactors. Using Crammers rule we still have to find the determinant of matrix A but we then only have to find the determinant of the three "A_i" matrices.

$$P_1 = \frac{\begin{vmatrix} 31 & -1 & -1 \\ 26 & 6 & -2 \\ 24 & -2 & 7 \end{vmatrix}}{\begin{vmatrix} 11 & -1 & -1 \\ -1 & 6 & -2 \\ -1 & -2 & 7 \end{vmatrix}} = 4, P_2 = \frac{\begin{vmatrix} 11 & 31 & -1 \\ -1 & 26 & -2 \\ -1 & 24 & 7 \end{vmatrix}}{\begin{vmatrix} 11 & -1 & -1 \\ -1 & 6 & -2 \\ -1 & -2 & 7 \end{vmatrix}} = 7, P_3 = \frac{\begin{vmatrix} 11 & -1 & 31 \\ -1 & 6 & 26 \\ -1 & -2 & 24 \end{vmatrix}}{\begin{vmatrix} 11 & -1 & -1 \\ -1 & 6 & -2 \\ -1 & -2 & 7 \end{vmatrix}} = 6$$

Or using the inversion approach:

x=A^{-1}b

$$x = \frac{1}{401} \begin{pmatrix} 38 & 9 & 8 \\ 9 & 76 & 23 \\ 8 & 23 & 65 \end{pmatrix} \begin{pmatrix} 31 \\ 26 \\ 24 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 6 \end{pmatrix}$$

Students should note however the relative merits of Crammers rule over the inversion approach only applies to manual calculations. The screen shot below shows how quick it was to find the solutions to these problems simply by using the inversion approach. The file is available online at: www.lawseconomics.co.uk/metal/example16.xls
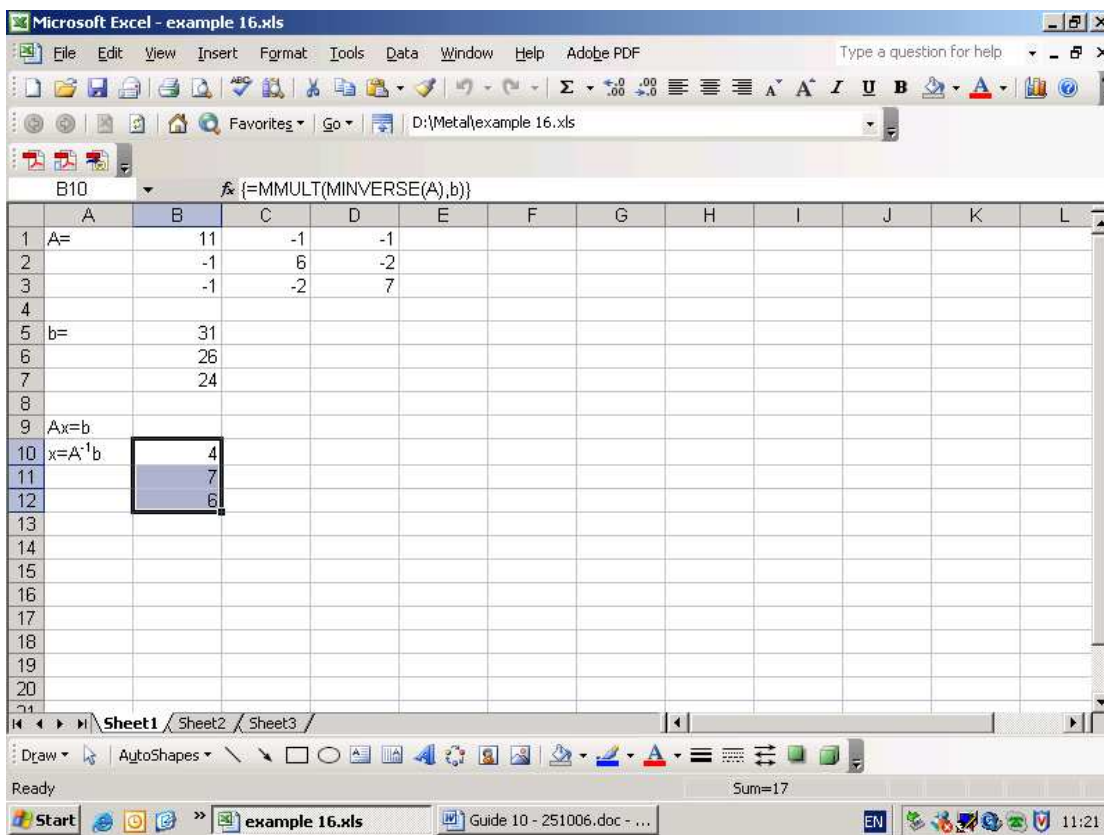
Links with the online question bank

See http://www.metalproject.co.uk/METAL/Resources/Question_bank/Matrices.html

4. Discussion Questions

Students could discuss why Cramer's rule offers an advantage in terms of calculation.

5. Activities

Learning Objectives

LO1: Students learn the practical application of Cramer's rule to national income determination

LO2: Students apply their knowledge to a simple problem of evaluating national income

Task One

Given the following typical National Income model found in most introductory macroeconomics text books:

$Y = C + I + G$

$C = bY_d$

$T = tY$

Where Y= Income, Y_d = disposable income, C = Consumption, T=Taxation, b = marginal propensity to consume, t = rate of taxation, I = autonomous Investment and G = autonomous Government spending.

Hence:

Re-writing the three equations in terms of Y, C and T we obtain:

$Y - C = I + G$

$C = b(Y-T) = bY - bT$ and so $-bY + C + bT = 0$

$-tY + T = 0$

These three equations can be represented in matrix form as:

$$\begin{pmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{pmatrix} \begin{pmatrix} Y \\ C \\ T \end{pmatrix} = \begin{pmatrix} I+G \\ 0 \\ 0 \end{pmatrix}$$

Using Crammers rule we can solve for the equilibrium levels of Y and/or C and/or T as follows:

$$Y = \frac{\begin{vmatrix} I+G & -1 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{(I+G)\begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 0 & b \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}}{1\begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + 0\begin{vmatrix} -b & 1 \\ -t & 0 \end{vmatrix}} = \frac{I+G}{1+bt-b}$$

$$C = \frac{\begin{vmatrix} 1 & I+G & 0 \\ -b & 0 & b \\ -t & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{1\begin{vmatrix} 0 & b \\ 0 & 0 \end{vmatrix} - (I+G)\begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + 0\begin{vmatrix} -b & 0 \\ -t & 0 \end{vmatrix}}{1\begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + 0\begin{vmatrix} -b & 1 \\ -t & 0 \end{vmatrix}} = \frac{(I+G)b(1-t)}{1+bt-b}$$

$$T = \frac{\begin{vmatrix} 1 & -1 & I+G \\ -b & 1 & 0 \\ -t & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{1\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - (-1)\begin{vmatrix} -b & 0 \\ -t & 0 \end{vmatrix} + (I+G)\begin{vmatrix} -b & 1 \\ -t & 0 \end{vmatrix}}{1\begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + 0\begin{vmatrix} -b & 1 \\ -t & 0 \end{vmatrix}} = \frac{(I+G)t}{1+bt-b}$$

We therefore have algebraic solutions that allow us to consider what would happen to the equilibrium levels of Y, C and T if, for example, the Government changed G or t.

Task One

(a) Review the above exposition of Cramer's rule.

(b) Assume that

I=300, G=200, b = 0.7 and t = 0.2.

What is the value of Y?

ANSWER

Task One

(b)

$$Y = \frac{I+G}{1+bt-b} = \frac{300+200}{1+0.7\times0.2-0.7} = \frac{500}{1+0.14-0.7} = \frac{500}{0.44} = 1136.36$$

$$C = \frac{(I+G)b(1-t)}{1+bt-b} = \frac{(200+300)\times0.7\times(0.8)}{0.44} = \frac{500\times0.56}{0.44} = 636.36$$

$$T = \frac{(I+G)t}{1+bt-b} = \frac{(200+300)\times0.2}{0.44} = \frac{100}{0.44} = 227.27$$

$$Y = C + I + G = 636.36 + 300 + 200 = 1136.36$$

6. Top Tips

Ask students to use Excel wherever they can to articulate and solve matrix algebra problems. Can they show their answer to 5. (see above) using Excel?

7. Conclusion

Students could be asked to create their own plenary as part of a conclusion on Cramer's rule. For example, how does Cramer's rule help us as economists? What sort of economic issues might it be particularly applicable to? This evaluation of Cramer's rule could be a good way to conclude this topic.

Section 5: Input-Output Analysis

1. The concept of input-output analysis

This material could be prefaced by explaining that in classical economic analysis it is assumed that a country's economic activity can be divided into a number of industries which produce goods and services. In addition, it is further assumed that the end product of all these industries work is final consumer demand.

This can then be linked to input-output analysis: it is further assumed that changes in demand for any of the final outputs from any industry affect the outputs that it requires from other industries.

2. Presenting the concept of input-output analysis

$$x_{ij} = a_{ij}x_j$$

Students will need a sound and formal explanation to which they can relate when attempting applied questions on their own. For example, students could be shown that we can articulate input-output analysis as follows:

Let: x_j be the total output of industry j,

x_{ij} be the output of industry i to industry j and a_{ij} is a constant.

If there are n industries under consideration:

$$x_i = \sum_{j=1}^{n} x_{ij} + y_i$$

Where y_i is the output of industry i to final demand. Students should note that this equation holds since total output of an industry must go either as inputs to other industries or to final demand.

Combining these two equations students can see that we obtain:

$$x_i = \sum_{j=1}^{n} a_{ij}x_j + y_i \Rightarrow x_i - \sum_{j=1}^{n} a_{ij}x_j = y_i$$

Expanding out the summation sign and consider for all values of i from 1 to n:

$$
\begin{array}{l}
x_1 \quad - a_{11}x_1 \quad - a_{12}x_2 \quad ... \quad - a_{1n}x_n = y_1 \\
x_2 \quad - a_{21}x_2 \quad - a_{221}x_3 \quad ... \quad - a_{2n}x_n = y_2 \\
\qquad .. \qquad\qquad\qquad\qquad\qquad ... \\
\qquad ... \qquad\qquad\qquad\qquad\qquad ... \\
\qquad ... \qquad\qquad\qquad\qquad\qquad ... \\
x_n \quad - a_{n1}x_1 \quad - a_{n2}x_2 \quad ... \quad a_{nn}x_n = y_n
\end{array}
$$

Clearly it is possible to group the x_1's, x_2's etc. together in a matrix form as follows:

$$
\begin{pmatrix}
1-a_{11} & -a_{12} & -a_{13} & ... & -a_{1n} \\
-a_{21} & 1-a_{22} & -a_{23} & ... & -a_{2n} \\
 & & & ... & \\
 & & & ... & \\
-a_{n1} & -a_{n2} & -a_{n3} & ... & 1-a_{nn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ ... \\ ... \\ x_n
\end{pmatrix}
=
\begin{pmatrix}
y_1 \\ y_2 \\ ... \\ ... \\ y_n
\end{pmatrix}
$$

If we create a matrix of input-output coefficients, A:

$$
A =
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & ... & a_{1n} \\
a_{21} & a_{22} & a_{23} & ... & a_{2n} \\
 & & & ... & \\
 & & & ... & \\
a_{n1} & a_{n2} & a_{n3} & ... & a_{nn}
\end{pmatrix}
$$

Then the matrix on the left is:

$$I - A$$

Where I is the identity matrix of order n. Students might want to revisit their earlier notes and material at this point.

Lecturers could highlight that we often call the column of total outputs X, and the column vector of outputs to final demand Y, then:

$$(I - A)X = Y$$

And hence:

$$X = (I-A)^{-1}Y$$

(assuming that the inverse of $(I - A)$ exists. This is an important result since if we are given the required outputs to final demand of each industry and provided we know the input-output coefficients we can obtain the required total outputs of each industry.

3. Delivering the concept of input-output analysis to small or larger groups
Worked example

Working national input-output models deal with many industries. However the basic principle can be illustrated with the aid of a small scale (n=3) example:

$$A = \begin{pmatrix} 1/8 & 1/3 & 1/4 \\ 1/2 & 1/6 & 1/4 \\ 1/4 & 1/6 & 1/4 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1001 \\ 3407 \\ 1505 \end{pmatrix}$$

Given A and Y we can find the output of each of the three industries as follows:

$$X = (I-A)^{-1}Y$$

In order to find the inverse of A we must first find I-A:

$$I - A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1/8 & 1/3 & 1/4 \\ 1/2 & 1/6 & 1/4 \\ 1/4 & 1/6 & 1/4 \end{pmatrix} = \begin{pmatrix} 7/8 & -1/3 & -1/4 \\ -1/2 & 5/6 & -1/4 \\ -1/4 & -1/6 & 3/4 \end{pmatrix}$$

And then find the determinant of the resultant matrix:

Expanding by column 1:

$$7/8 \begin{vmatrix} 5/6 & -1/4 \\ -1/6 & 3/4 \end{vmatrix} - (-1/2) \begin{vmatrix} -1/3 & -1/4 \\ -1/6 & 3/4 \end{vmatrix} + (-1/4) \begin{vmatrix} -1/3 & -1/4 \\ 5/6 & -1/4 \end{vmatrix}$$

$$= \frac{7}{8} \times \left(\frac{15}{24} - \frac{1}{24} \right) + \frac{1}{2} \times \left(-\frac{3}{12} - \frac{1}{24} \right) - \frac{1}{4} \times \left(\frac{1}{12} + \frac{5}{24} \right)$$

$$= \frac{7}{8} \times \left(\frac{7}{12} \right) + \frac{1}{2} \times \left(-\frac{7}{24} \right) - \frac{1}{4} \times \left(\frac{7}{24} \right)$$

$$= +49/96 - 7/48 - 7/96 = -70/96 = 7/24$$

The finding the nine cofactors:

$$A_{11} = +(15/24 - 1/24) = 7/12$$
$$A_{21} = -(-3/12 - 1/24) = 7/24$$
$$A_{31} = +(1/12 + 5/24) = 7/24$$
$$A_{12} = -(-3/8 - 1/16) = 7/16$$
$$A_{22} = +(21/32 - 1/16) = 19/32$$
$$A_{32} = -(-7/32 - 1/8) = 11/32$$
$$A_{13} = +(1/12 + 5/24) = 7/24$$
$$A_{23} = -(-7/48 - 1/12) = 11/48$$
$$A_{33} = +(35/48 - 1/6) = 27/48 = 9/16$$

Putting these in-order forms the <u>adjugate</u> matrix:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} \frac{7}{12} & \frac{7}{16} & \frac{7}{24} \\ \frac{7}{24} & \frac{19}{32} & \frac{11}{48} \\ \frac{7}{24} & \frac{11}{32} & \frac{9}{16} \end{pmatrix}$$

Then taking the transpose of the adjugate matrix to form the <u>adjoint</u> matrix:

$$\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} \frac{7}{12} & \frac{7}{24} & \frac{7}{24} \\ \frac{7}{16} & \frac{19}{32} & \frac{11}{32} \\ \frac{7}{24} & \frac{11}{48} & \frac{9}{16} \end{pmatrix}$$

The inverse is therefore:

$$(I-A)^{-1} = \frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \frac{24}{7}\begin{pmatrix} \frac{7}{12} & \frac{7}{24} & \frac{7}{24} \\ \frac{7}{16} & \frac{19}{32} & \frac{11}{32} \\ \frac{7}{24} & \frac{11}{48} & \frac{9}{16} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 3/2 & 59/29 & 86/73 \\ 1 & 73/93 & 27/14 \end{pmatrix}$$

Multiplying this by the column vector of final demand we find approximately:

$$x = \begin{pmatrix} 2 & 1 & 1 \\ 3/2 & 59/29 & 86/73 \\ 1 & 73/93 & 27/14 \end{pmatrix}\begin{pmatrix} 1001 \\ 3407 \\ 1505 \end{pmatrix} = \begin{pmatrix} 6907 \\ 10205 \\ 6577 \end{pmatrix}$$

Which gives directly the outputs given (or required) from each industry as being 6907, 10205 and 6577 respectively.

Note that we can also calculate the vector X-Y which gives directly that output of each industry that goes to other industries rather than as final demand.

Worked example: Extension Material

Finally, from the equation:

$$X = (I-A)^{-1}Y$$

We can find the marginal effect of having to produce one extra unit of final demand:

$$\square X = (I-A)^{-1}\square Y$$

Where $\square X$ and $\square Y$ are the respective small increases. Hence, if in the example above we needed to produce one extra unit for industry 3 the $\square Y$ vector would be:

$$\Delta Y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and therefore:

$$\Delta X = (I - A)^{-1} \Delta Y = \begin{pmatrix} 2 & 1 & 1 \\ 3/2 & 59/29 & 86/73 \\ 1 & 73/93 & 27/14 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 86/73 \\ 27/14 \end{pmatrix}$$

Links with the online question bank

Questions focused on matrices can be found at

http://www.metalproject.co.uk/METAL/Resources/Question_bank/Matrices.html

4. Discussion Questions

Students could be asked to find real world economic stories from quality news sources e.g. *The Economist* to show how input-output analysis could be applied. For example, what knock-on might the demise of the Rover car company have on other industries and the UK economy?

5. Activities

Task One

Learning Objectives

LO1: Students learn how to apply their knowledge of matrices to input-output analysis

LO2: Students learn how to draw inferences from answers derived using matrix algebra

Task One

Ask students to review the worked example above (See 3.). Working in pairs, students should create a Powerpoint presentation which summarises:

- what the worked example was trying to solve

- the **key** steps to solving the input-output problem; and

- any problems and difficulties they encountered when working through the worked example.

Task Two

Working in pairs. create your own values of A and Y and assume n=3.

6. Top Tips

Students will want to practise questions having first ensured they have mastered each of the steps (See above).

Lecturers will probably want to create opportunities for students to Excel to apply their knowledge and understanding. Higher ability students could be asked to develop their own problem sets and answers.

7. Conclusion

This is very much a summative or plenary topic. Create chances for students to work through problem sets and ideally offer one-to-one support as they design and create their own independent input-output problem set (See 5. above).